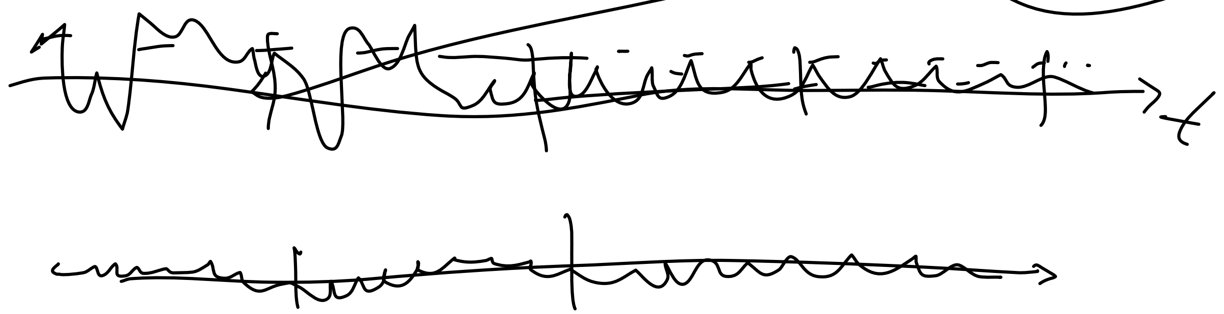
Lezione 1

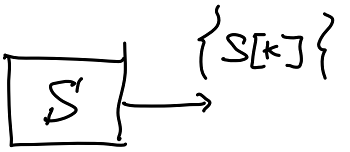
lunedì 5 ottobre 2020 10.55

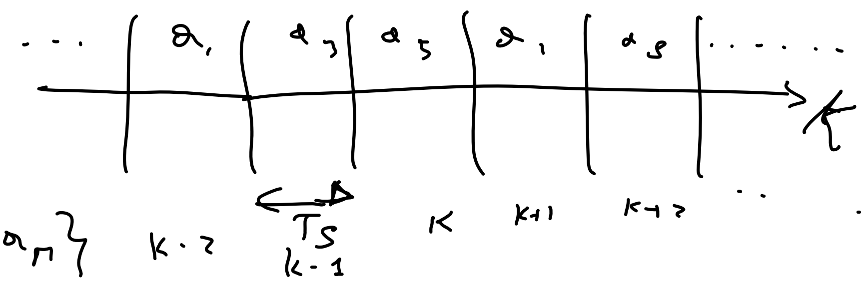
· · · · 1 0 1 0 0 1 | 0 | 0 | 1 | 1 | 0 | 0 · · · · ·

$\underbrace{0\ 0}_{T_s}$ $\underbrace{1\ 0}_{\text{freq. relativa}}$

$\sqrt{P_s}$ $P_s(f)$







$S[k] \in \mathcal{A} = \{a_1, a_2, \dots, a_M\}$

Es: $M = 2$ $\mathcal{A} = \{0,1\}$ $\mathcal{A} = \{A,B\}$

$M = 3$ $\mathcal{A} = \{0,1,*\}$

0 1 * 0 1 * * 1 1 * 0 · · · ·

$M = 8$ $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5 \dots a_8\}$

{000, 001, 010, · · · · · · · · , 111}

000 ⋮ 010 ⋮ 111 ⋮ · · · · · }

$\xleftrightarrow{T_s}$

BAUDRATE Freq. di simbolo $\frac{1}{T_s}$ simboli/sec.

TASSO BINARIO
BIT-RATE $B_r = \frac{\log_2 M}{T_s}$ bit/sec

0 1 0 0 1 · 0 0 0 0 0 1 0 0 1

cont inf. diverso? ↕

$\overline{\quad}^{\,}_{T_s}$

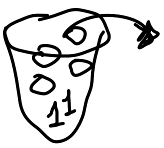
$T_s = 1 \mu sec$

$B_r = \frac{1}{1} \frac{bit}{} = 10^6 \, bit$

diverso? ⇓ $_s$ $_s = 1 \mu sec$

0 1 1 0 0 1 1 0 1 0 $B_r = \frac{1}{T_s} \frac{bit}{sec} = 10^6 \frac{bit}{sec}$

$= 1 \, Mbit/sec$



$\mathcal{A} = \{a_1 \; a_2 \dots a_M\}$

$\Pi = \{p_1, p_2 \dots p_M\}$

$0 \le p_i \le 1$

$\sum_{i=1}^{M} p_i = 1$

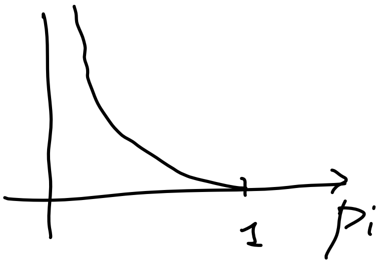
Es. $M = 3$ $\mathcal{A} = \{a_1 \; a_2 \; a_3\}$

$\Pi = \{0.001, 0.5, 0.499\}$

$$B_r = \frac{\log_2 3}{T_s}$$

$$H(S) = E\left[\log_2 \frac{1}{p_i}\right] = \sum_{i=1}^{M} p_i \log_2 \frac{1}{p_i}$$

ENTROPIA bit



Autoinformazione.

$\log_3$ → trit

$\ln$ → nat

$B_r = \frac{\log_2 M}{T_s}$ FREQ. DI CIFRA $R_s = \frac{H(S)}{T_s}$ TASSO ENTROPICO bit/sec

⟷

$\mathcal{A} = \{a_1 \dots a_M\}$

$\Pi = \{\frac{1}{M}, \dots \frac{1}{M}\}$

$H(S) = \sum_{i=1}^{M} p_i \log_2 \frac{1}{p_i} = \sum_{i=1}^{M} \frac{1}{M} \log_2 M = \log_2 M$

$$\boxed{B_r = R_s}$$

CODICE MORSE

E . T —

I .. M — —

S ... O — — —

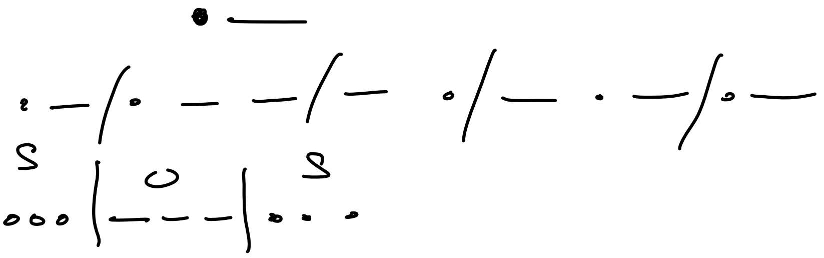
H .... CH — — — —

A .— N —.

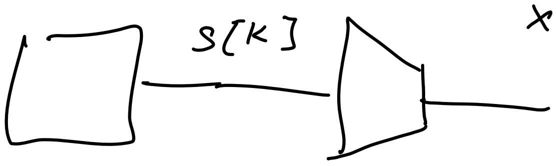
A B C D E - - - — . . . .

● —

/ / / /



A •– N –•
U ••– D –••
V •••– B –•••



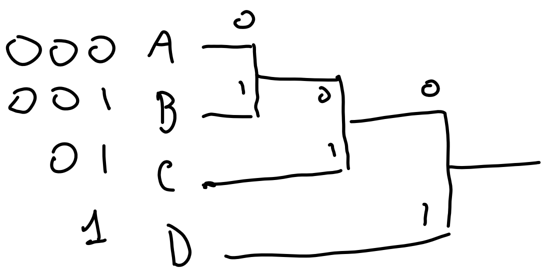
x[K] → lunghezza variabile contenuti da un altro alfabeto base. •–

S[K] = S = {a, ... am}

{0,1}

a_1	00
a_2	011
:	0110
a_m	011·10

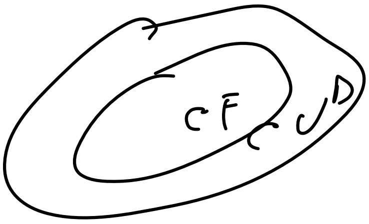
ABCD



ACDAAC

000 01 1 00000001

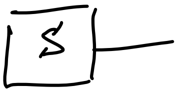
A C D A A C



CODICE A PREFISSO

Lezione 2

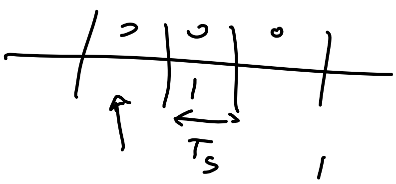
9 ottobre ore 11:00



$S[k] \in \{a_1, \dots, a_M\}$

$\pi = \{p_1 \cdots p_M\}$

$$H(S) = \sum_{i=1}^{M} p_i \log_2 \frac{1}{p_i}$$



$\frac{1}{T_s}$ baud

$B_2 = \frac{\log_2 M}{T_s}$ bit/sec

$R_s = \frac{H(S)}{T_s}$

$$H(S) \le \log_2 M$$

$$p_1 = p_2 = \dots = p_M = \frac{1}{M} \Longrightarrow H(S) = \log_2 M$$

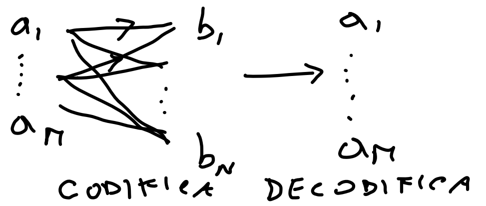
Sorgente binaria $A = \{0, 1\}$ $\pi = \{0.1, 0.9\}$

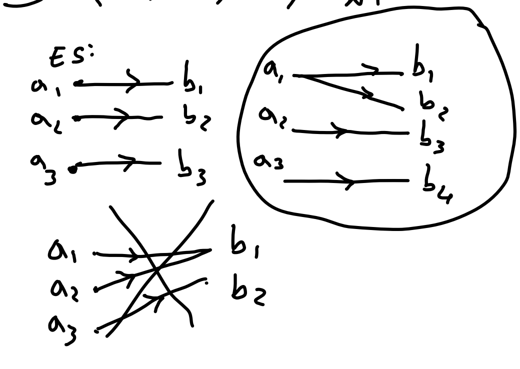
$$0\;0\;\underbrace{1\;0\;0\;1\;1\;1}\;0\;1\;1\;1\;0\;1\;1\;0$$

$$H(S) = 0.1 \log_2 \frac{1}{0.1} + 0.9 \log_2 \frac{1}{0.9} = 0.47 \text{ bit}$$

CODICE UNIVOCAMENTE DECODIFICABILE

$$A = \{a_1 \dots a_n\} \longleftrightarrow B = \{b_1, b_2, \dots, b_N\}$$





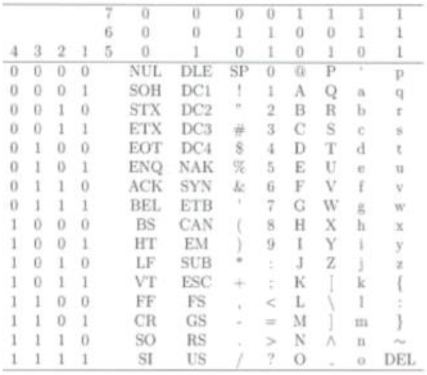
Stringa di simboli $\in$ Alfabeto base C

Lunghezza fissa $n_i = n$ $\forall i$

lunghezza variabile

Lunghezza fissa $n_i = n \quad \forall i$

Lunghezza variabile



				7	0	0	0	0	1	1	1	1
				6	0	0	1	1	0	0	1	1
4	3	2	1	5	0	1	0	1	0	1	0	1
0	0	0	0		NUL	DLE	SP	0	@	P	'	p
0	0	0	1		SOH	DC1	!	1	A	Q	a	q
0	0	1	0		STX	DC2	"	2	B	R	b	r
0	0	1	1		ETX	DC3	#	3	C	S	c	s
0	1	0	0		EOT	DC4	$	4	D	T	d	t
0	1	0	1		ENQ	NAK	%	5	E	U	e	u
0	1	1	0		ACK	SYN	&	6	F	V	f	v
0	1	1	1		BEL	ETB	'	7	G	W	g	w
1	0	0	0		BS	CAN	(	8	H	X	h	x
1	0	0	1		HT	EM	)	9	I	Y	i	y
1	0	1	0		LF	SUB	*	:	J	Z	j	z
1	0	1	1		VT	ESC	+	;	K	[	k	{
1	1	0	0		FF	FS	,	<	L	\	l	:
1	1	0	1		CR	GS	-	=	M	]	m	}
1	1	1	0		SO	RS	.	>	N	∧	n	~
1	1	1	1		SI	US	/	?	O	.	o	DEL

Tabella 3.1: Il codice ASCII

$C = \{0, 1\}$

Lunghezza fissa $M = 128$

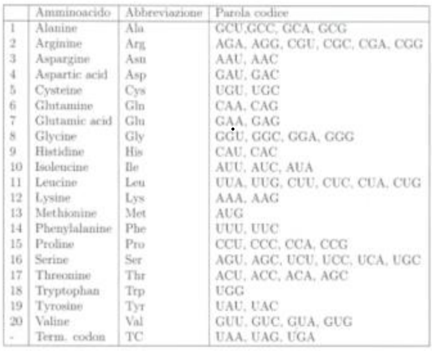
$n = 7$

A → 1000 001 ⋮ 1

G → 1110 001 ⋮ 1

bit di parità

$N = 128$



	Amminoacido	Abbreviazione	Parola codice
1	Alanine	Ala	GCU,GCC, GCA, GCG
2	Arginine	Arg	AGA, AGG, CGU, CGC, CGA, CGG
3	Aspargine	Asn	AAU, AAC
4	Aspartic acid	Asp	GAU, GAC
5	Cysteine	Cys	UGU, UGC
6	Glutamine	Gln	CAA, CAG
7	Glutamic acid	Glu	GAA, GAG
8	Glycine	Gly	GGU, GGC, GGA, GGG
9	Histidine	His	CAU, CAC
10	Isoleucine	Ile	AUU, AUC, AUA
11	Leucine	Leu	UUA, UUG, CUU, CUC, CUA, CUG
12	Lysine	Lys	AAA, AAG
13	Methionine	Met	AUG
14	Phenylalanine	Phe	UUU, UUC
15	Proline	Pro	CCU, CCC, CCA, CCG
16	Serine	Ser	AGU, AGC, UCU, UCC, UCA, UGC
17	Threonine	Thr	ACU, ACC, ACA, AGC
18	Tryptophan	Trp	UGG
19	Tyrosine	Tyr	UAU, UAC
20	Valine	Val	GUU, GUC, GUA, GUG
-	Term. codon	TC	UAA, UAG, UGA

Tabella 3.2: Il codice genetico

Lunghezza fissa $n = 3$

$C = \{AGCU\}$ $\quad 4^3 = 64$

Ass. Amm. → Stringa
non univoca

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$
A	· —	1	· — — — —	.	· — · — · —
B	— · · ·	2	· · — — —	,	— — · · — —
C	— · — ·	3	· · · — —	?	· · — — · ·
D	— · ·	4	· · · · —	"	· — · · — ·
E	·	5	· · · · ·	:	— — — · · ·
F	· · — ·	6	— · · · ·	;	— · — · — ·
G	— — ·	7	— — · · ·	)	— · — — · —
H	· · · ·	8	— — — · ·	(	— · — — ·
I	· ·	9	— — — — ·	'	· — — — — ·
J	· — — —	0	— — — — —	Double dash (Break)	— · · · —
K	— · —			Error	· · · · · · · ·
L	· — · ·			/	— · · — ·
M	— —			End of message (AR)	· — · — ·
N	— ·			End of Transmission (SK)	· · · — · —
O	— — —			SOS	· · · — — — · · ·
P	· — — ·			Wait (AS)	· — · · ·
Q	— — · —				
R	· — ·				
S	· · ·				
T	—				
U	· · —				
V	· · · —				
W	· — —				
X	— · · —				
Y	— · — —				
Z	— — · ·				

Tabella 3.3: Il codice Morse

Lunghezza Variabile

$C = \{·, —\}$

FEDE

· · — · | · | — · · | ·

E A I N E

UNIVOCAMENTE DECODIFICABILI → Seq. simboli di sorgente è univoca

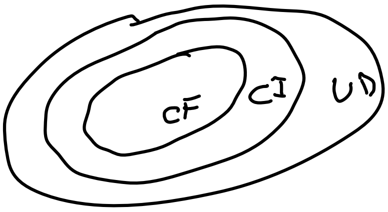
a_1 00

a_2 111

$a_1\ a_2\ a_1\ a_1$

00 111 00 00

CODICE ISTANTANEO



CODICI A PREFISSO: nessuna parola codice è prefisso di un'altra

PAROLE CODICE ⟷ TERMINAZIONI DI UN ALBERO

ARIO o

PAROLE CODICE ⟷ TERMINAZIONI DI UN ALBERO

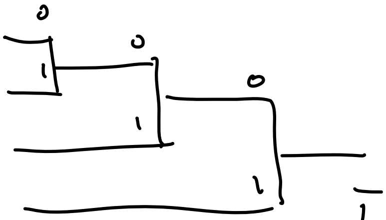
BINARIO

0.1 000 a_1

0.2 001 a_2

0.5 01 a_3

0.2 1 a_4



$\underbrace{001}_{a_2},\underbrace{1}_{a_4},\underbrace{001}_{a_2},\underbrace{000}_{a_1}$

$\bar{L} = 3\cdot 0.1 + 3\cdot 0.2 + 2\cdot 0.5 + 0.2$
$= 2.1$ bit

TERNARIO $C = \{A,B,C\}$

0.1 AAA a_1

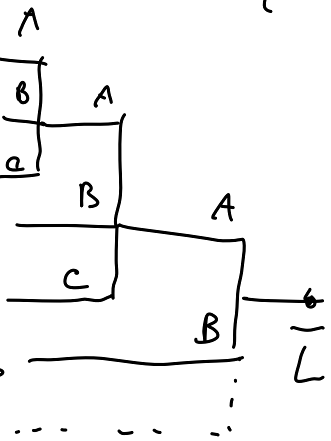
0.2 AAB a_2

0.5 AAC a_3

0.1 AB a_4

0.05 AC a_5

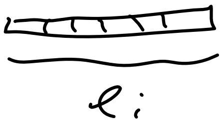
0.05 B a_6



$\underbrace{AB}_{a_4}\underbrace{B}_{a_6}\underbrace{AC}_{a_5}\underbrace{AB}_{a_4}$

$\bar{L} = 3\cdot 0.1 + 3\cdot 0.2 + 3\cdot 0.5 + 2\cdot 0.1 + 2\cdot 0.05 +$
$+ 0.05$
$= X$

$a_i \longrightarrow$



p_i

$C = \{0,1\}$

$$\bar{L} = \sum_{i=1}^{M} p_i l_i$$

$\{a_1, a_2, a_3, a_4, a_5\}$
$\{0.1, 0.3, 001, 0.5, 0.09\}$

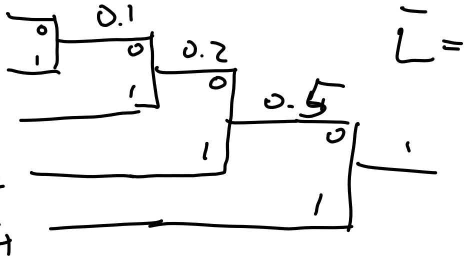
0000 0.01 a_3

0001 0.09 a_5

001 0.1 a_1

01 0.3 a_2

1 0.5 a_4



$\bar{L} = 4\cdot 0.01 + 4\cdot 0.09 + 3\cdot 0.1$
$+ 2\cdot 0.3 + 0.5 = X$ bit

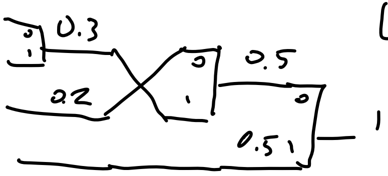
$\{a, a_2\ a_3\ a_4\}$
$\{0.2\ 0.1\ 0.2, 0.5\}$

010 0.1 a_2

011 0.2 a_1

00 0.2 a_3

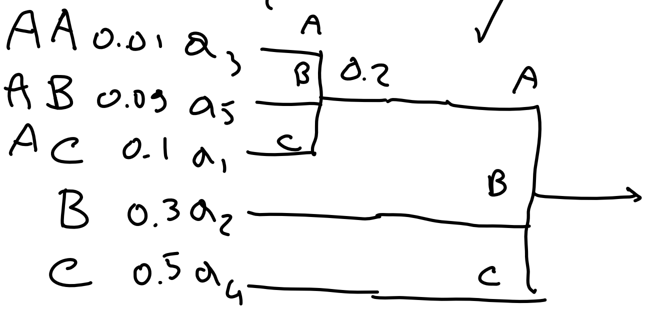
1 0.5 a_4



$\bar{L}$. - - - -

$\{a_1, a_2, a_3, a_4, a_5\}$

$\{0.1, 0.3, 001, 0.5, 009\}$



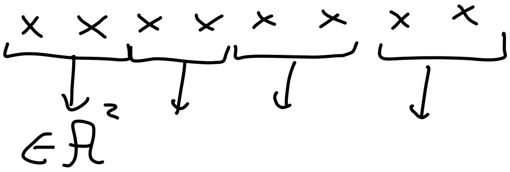
l_i

$$\boxed{\sum_{i=1}^{n} 2^{-l_i} \leq 1}$$

DIS. DI KRAFT-McMILLAN → Cond. nec. per C.P.

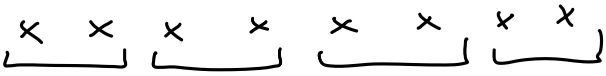
↓ 3 nel caso ternario

$\boxed{S} \rightarrow$ $s[k] \in \mathcal{A} = \{a_1 \ldots a_m\}$



SORGENTE
ESTESA

$\mathcal{A} = \{a_1, a_2, a_3\}$



ordine 2

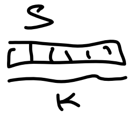
$\mathcal{A}^2 = \{a_1a_1, a_1a_2, a_1a_3, a_2a_1, a_2a_2, a_2a_3, a_3a_1, a_3a_2, a_3a_3\}$

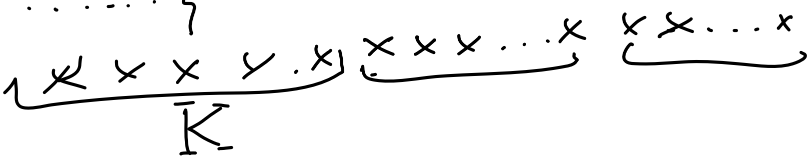
$= \mathcal{A} \times \mathcal{A}$ $\qquad \boxed{S^2} \rightarrow s^2 \in \mathcal{A}^2$

$\mathcal{A}^3 = \underbrace{\mathcal{A} \times \mathcal{A}} \times \mathcal{A} = \mathcal{A}^2 \times \mathcal{A}$

$= \{a_1a_1a_1, a_1a_1a_2, a_1a_1a_3, \ldots\ldots\}$

$\boxed{S^K} - s^K$





$s^K \in \mathcal{A}^K = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{K}$

$\mathcal{A} = \{a_1\, a_2\, a_3\}$

$\Pi = \{p_1, p_2, p_3\}$

$$A^2 = \{a_1a_1, a_1a_2, a_1a_3, a_2a_1, a_2a_2, a_2a_3, a_3a_1, a_3a_2, a_3a_3\}$$

$$\Pi^2 = \{p_1^2, p_1p_2, p_1p_3, p_2p_1, p_2^2, p_2p_3, p_3p_1, p_3p_2, p_3^2\}$$

$$= \{p_1, p_2, p_3\} \otimes \{p_1 p_2 p_3\} = \{p_1\{p_1 p_2 p_3\}, p_2\{p_1 p_2 p_3\}, p_3\{p_1 p_2 p_3\}\}$$

$A = \{a_1, a_2, \ldots, a_M\}$ M-ario

$\Pi = \{p_1\ p_2\ \ldots\ p_M\}$

$$S^K \quad s^{\vec{K}} \in \underbrace{A \times A \times \cdots \times A}_{K} = A^K$$

$$\Pi^k = \underbrace{\Pi \otimes \Pi \otimes \cdots \otimes \Pi}_{K}$$

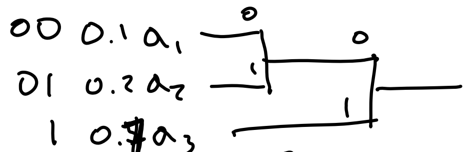
Esempio

$A = \{a_1\ a_2\ a_3\}$ $H(\Pi) = 1.1568$ bit

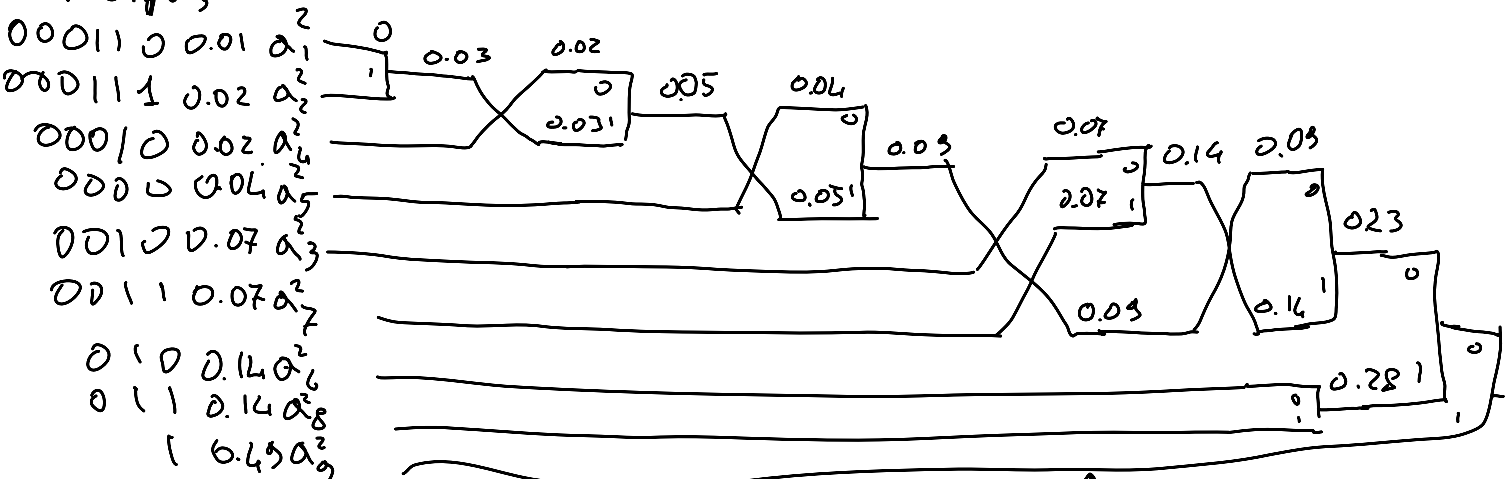
$\Pi = \{0.1, 0.2, 0.7\}$

$$A^2 = \{\overset{a_1^2}{a_1a_1}, \overset{a_2^2}{a_1a_2}, \overset{a_3^2}{a_1a_3}, \overset{a_4^2}{a_2a_1}, \overset{a_5^2}{a_2a_2}, \overset{a_6^2}{a_2a_3}, \overset{a_7^2}{a_3a_1}, \overset{a_8^2}{a_3a_2}, \overset{a_9^2}{a_3a_3}\}$$

$\Pi^2 = \Pi \otimes \Pi =$ ~~0.0100~~ ~~0.0200~~ ~~0.0700~~ ~~0.0200~~ ~~0.0400~~ ~~0.1400~~ ~~0.0700~~ 0.1400 0.4900



$\bar{L} = 2 \cdot 0.1 + 2 \cdot 0.2 + 0.7 = 1.3$ bit



1	2	3	4	5	6	7	8	9
6	6	4	5	4	3	4	3	1

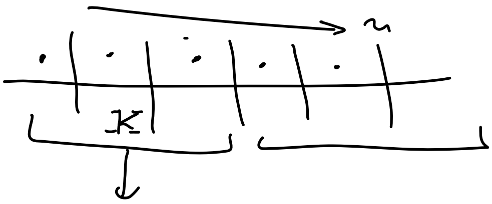
$\bar{L} = 2.33$ bit

$$\bar{L}_2 = \frac{\bar{L}}{2} = 1.16 \text{ bit}$$

$\boxed{S} \to s[n]$

$s[n] \in \mathcal{A} = \{a_1 \dots a_n\}$

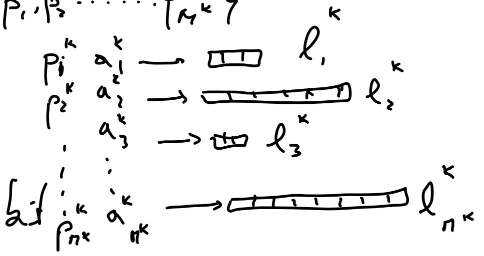
$\pi = \{p_1 \cdots p_n\}$



$$S^K \to s^k \in \mathcal{A}^k = \underbrace{\mathcal{A} \times \mathcal{A} \times \quad \times \mathcal{A}}_{k} = \{a_1^k, a_2^k, \dots, a_{n^k}^k\}$$

$$\pi^k = \pi \otimes \pi \otimes \dots \otimes \pi = \{p_1^k, p_2^k \cdots p_{n^k}^k\}$$

$S^k \longrightarrow$



$$\bar{L}_K = \sum_{i=1}^{n^k} p_i^k l_i^k$$

$$\frac{\bar{L}_K}{K} \xrightarrow{?} H(\pi)$$

$$l_i^K = \left\lceil \lg_2 \frac{1}{p_i^K} \right\rceil \qquad \lg_2 \frac{1}{p_i^K} < l_i^K < \lg_2 \frac{1}{p_i^K} + 1$$

Dis di Kraft
Soddisfatta

$$\boxed{\sum_{i=1}^{N} 2^{-l_i} \leq 1}$$

$$\boxed{\lg_2 \frac{1}{p_i^k} < l_i^k < \lg_2 \frac{1}{p_i^k} + 1}$$

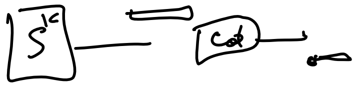
$$p_i^k \lg_2 \frac{1}{p_i^k} \leq p_i^k l_i^k < p_i^k \lg_2 \frac{1}{p_i^k} + p_i^k$$

$$\sum_i$$

$$\sum^{n^K} p_i^k l \frac{1}{p_i^k} \quad \sum_{i=1}^{n^k} \cdot l_i^k < \sum_{i=1}^{n^k} p_i^k \lg_2 \frac{1}{p_i^k} + 1$$

$$\boxed{\frac{H(S^K)}{K} < \frac{\bar{L}_K}{K} < \frac{H(S^K)}{K} + \frac{1}{K}}$$

$$K \to \frac{L_K}{K} \to \frac{H(S^K)}{K} = \frac{\cancel{K} H(S)}{\cancel{K}}$$



I° teorema

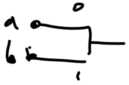
Esempio Sorgente binaria

$\mathcal{A} = \{a, b\}$ $\quad H(S) = 0.1 \lg_2 \frac{1}{0.1} + 0.9 \lg_2 \frac{1}{0.9} = 0.4690$

$\pi = \{0.1, 0.9\}$

$\mathcal{A} = \{a, b\}$ $\quad H(S) = 0.1 \log_2 \frac{1}{0.1} + 0.9 \log_2 \frac{1}{0.9} = 0.4690$

$\Pi = \{0.1, 0.9\}$



$\overline{L}_1 = 1 \cdot 0.1 + 1 \cdot 0.9 = 1$ bit

$\mathcal{A}^2 = \{aa, ab, ba, bb\} = \mathcal{A} \times \mathcal{A}$

$\Pi^3 = \Pi \otimes \Pi = \{0.01, 0.09, 0.09, 0.81\}$

$-\log_2 p_i \quad \{6.64, 3.47, 3.47, 0.3\}$

$\{7 \quad 4 \quad 4 \quad 1\}$

aa	0000000
ab	0011
ba	0111
bb	1

$\overline{L}_2 = 7 \cdot 0.01 + 4 \cdot 0.09 + 4 \cdot 0.09 + 1 \cdot 0.81 = 1.6$ bit

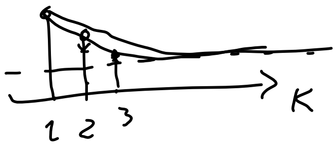
$\frac{\overline{L}_2}{2} = 0.8$

$\mathcal{A}^3 = \mathcal{A} \times \mathcal{A}^2 = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$

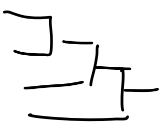
$\Pi^3 = \{$ 0.0010 0.0090 0.0090 0.0810 0.0090 0.0810 0.0810 $\bigcirc$0.7290 $\}$ $\}$

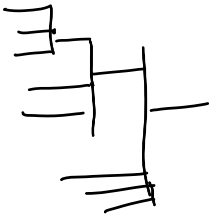
$-\log_2 \downarrow = \{$ 10 7 7 4 7 4 4 1 $\}$

$\overline{L}_3 = 1.9$ $\qquad \frac{\overline{L}_3}{3} = 0.63$



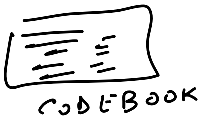
$$\sum_{i=1}^{N} 2^{-l_i} \leq 1 \qquad \boxed{\sum_{i=1}^{N} M^{-l_i} \leq 1}$$



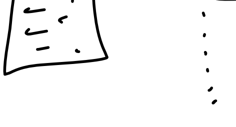


$\Pi^K = \{p_1^K, p_2^K \ldots p_{M^K}^K\}$ $\qquad H(S^k) = \cancel{K H(S)}$

HUFFMAN →



a a
a b
a u



→ CODICE

a
b
c
d
e

δ

LEMPEL-ZIV (ZIP)

Ω = {a b c}

INPUT	OUTPUT	CODEBOOK
bcababbcbcbaaaabbc		$\{a = 1, b = 2, c = 3\}$

b : c : a : b : a b : b c : b c b : a : a a : a b b : c

2 : 3 : 1 : 2 : 6 : 4 : 9 : 1 : 11 : 8 : 3

1	a
2	b
3	c
4	bc
5	ca
6	ab
7	ba
8	abb
9	bcb
10	bcba
11	aa
12	aaa
13	abbc

2 3 1 2 6 4 9 1 11 8 3

b c a b ab bc : bcb | a | aa | abbc

1	a
2	b
3	c
4	bc
5	ca
6	ab
7	ba
8	abb
9	bcb
10	bcba
11	aa
12	aaa

Lezione 4

venerdì 16 ottobre 2020 11.31

SEQUENZE TIPICHE

$\mathcal{A} = \{0, 1\}$

$\pi = \left\{\frac{8}{10}, \frac{2}{10}\right\}$

0 0 0 1 0 0 1 1 1 0 0 0 1

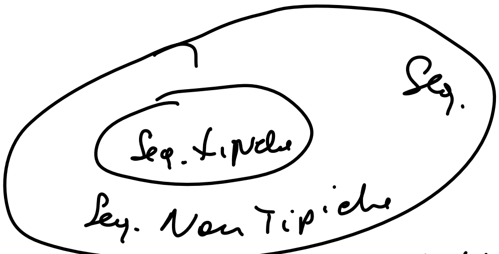
0 1 0 1 0 0 0 1 0 0 0 0

0 0 1 0 0 0 0 0 1 0 0 1 $\left(\frac{8}{10}\right)^{10} \cdot \left(\frac{2}{10}\right)^{3}$

$\rightarrow$ 1 1 1 1 1 1 1 1 1 1 1 1 1 $\left(\frac{2}{10}\right)^{13}$

$\rightarrow$ 1 0 0 1 1 1 1 1 0 0 1 1 0

Prob $\rightarrow 0$



\# seq. tipiche $\simeq 2^{nH(\pi)}$
di lunghezza n
(simboli indipendenti)

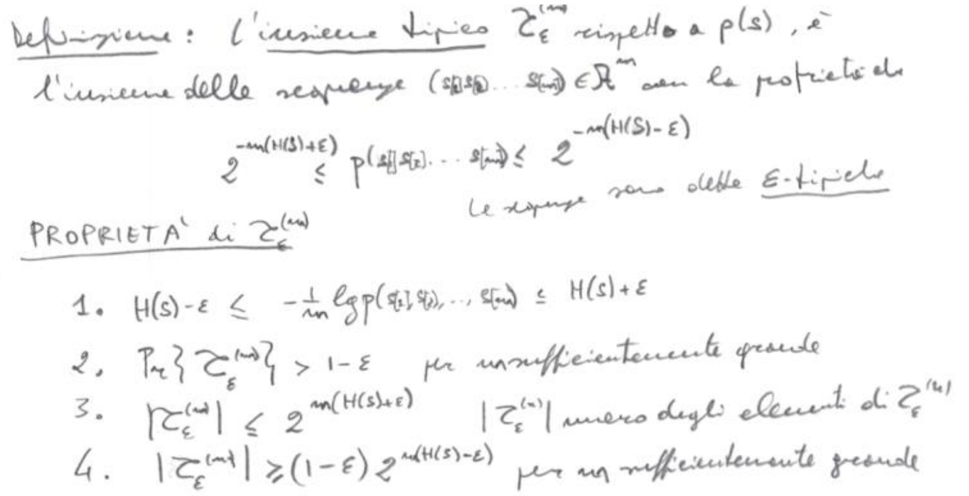
Es. caso binario

$\pi = \{p\ (1-p)\}$

$H(\pi) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p}$

$p = 1/2 \Rightarrow H(\pi) = 1$

seq. tipiche $= 2^n$



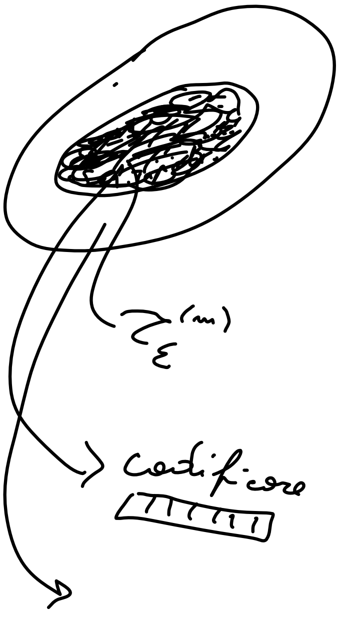
Definizione: l'insieme tipico $\mathcal{T}_\varepsilon^{(n)}$ rispetto a $p(s)$, è l'insieme delle sequenze $(s[1]s[2] \dots s[n]) \in \mathcal{A}^n$ con la proprietà che

$$2^{-n(H(S)+\varepsilon)} \leq p(s[1]s[2] \dots s[n]) \leq 2^{-n(H(S)-\varepsilon)}$$

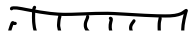
Le sequenze sono dette ε-tipiche

PROPRIETÀ di $\mathcal{T}_\varepsilon^{(n)}$

1. $H(S) - \varepsilon \leq -\frac{1}{n} \lg p(s[1], s[2], \dots, s[n]) \leq H(S) + \varepsilon$
2. $P_r\{\mathcal{T}_\varepsilon^{(n)}\} > 1 - \varepsilon$ per n sufficientemente grande
3. $|\mathcal{T}_\varepsilon^{(n)}| \leq 2^{n(H(S)+\varepsilon)}$ $|\mathcal{T}_\varepsilon^{(n)}|$ numero degli elementi di $\mathcal{T}_\varepsilon^{(n)}$
4. $|\mathcal{T}_\varepsilon^{(n)}| \geq (1-\varepsilon)\, 2^{n(H(S)-\varepsilon)}$ per n sufficientemente grande

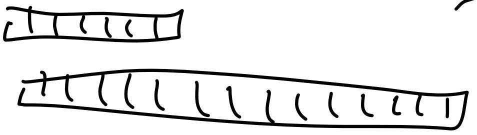


seq. tipiche



seq. tipiche

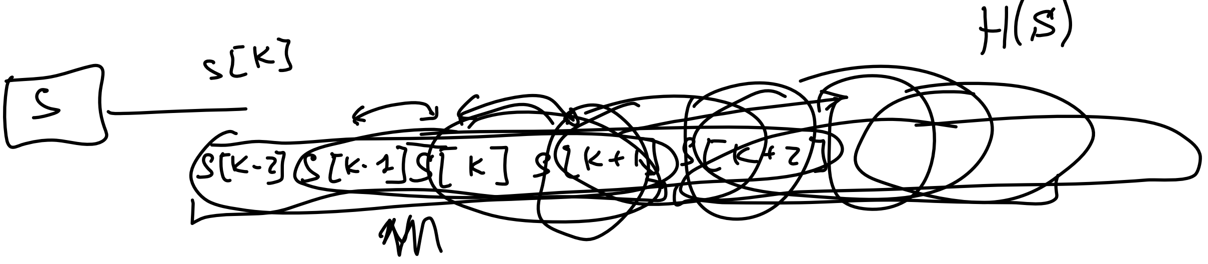
seq. non tipiche



$$\bar{L} \leq n H(\pi)$$

$$\frac{\bar{L}}{n} \rightarrow H(\pi)$$

SORGENTI CON MEMORIA



$$\underbrace{x_1 x_2 x_3 x_4 x_5 x_6}$$

$$P_{X_1 X_2 X_3 X_4 X_5 X_6}(x_1 x_2 x_3 x_4 x_5 x_6)$$

Staz. di ordine m

$$S^m = (S[k]\, S[k+1] \ldots\ldots S[k+m-1])$$

$$P_{S^m}(s_0 s_1 \ldots\ldots s_{m-1})$$

$$H(S^m) = \sum_{\substack{s_0 s_1 \ldots s_{m-1} \\ \in A^m}} P_{S^m}(s_0 \ldots s_{m-1}) \log_2 \frac{1}{P_{S^m}(s_0 \ldots s_{m-1})} \text{ bit.}$$

$$XY \quad f_{XY}(x,y)$$

$$\boxed{H(S) = \lim_{m\to\infty} \frac{H(S^m)}{m}}$$

TASSO ENTROPICO $\rightarrow \frac{m\, H(S)}{m}$

$$S[1]\, S[2] \ldots\ldots S[n] \rightarrow$$

$$H_\infty(S) = \lim_{n\to\infty} H\big(S[n] \,\big|\, S[n-1] \ldots\ldots S[1]\big) \text{ bit}$$

$$\boxed{\text{Sorg. stazionarie} \Rightarrow H_\infty(S) = H(S)}$$

$n \to \infty$

$$\boxed{\text{Seq. stazionarie} \Rightarrow H_\infty(S) = H(S)}$$

$x \in \mathcal{X} \quad y \in \mathcal{Y}$

$p(y|x)$

$p(x)$

$p(x,y)$

$$H(Y|X) \triangleq \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \underbrace{p(x,y)}_{\boxed{p(y|x)\,p(x)}} \lg_2 \frac{1}{p(y|x)}$$

$$H(Y|X) \leq H(Y)$$

$$H(S[2]|S[1]) \leq H(S[2])$$

REGOLA A CATENA

$$p(x_1 x_2) = p(x_2|x_1)\,p(x_1)$$

$$p(x_1 x_2 x_3) = p(x_3|x_1 x_2)\,p(x_1 x_2) = p(x_3|x_1 x_2)\,p(x_2|x_1)\,p(x_1)$$

$$H(X_1 X_2 X_3) = H(X_3|X_1 X_2) + H(X_2|X_1) + H(X_1)$$

$$\underbrace{H(S[1]\dots S[m])}_{m} = \frac{1}{m}\sum_{i=1}^{m} \underbrace{H(S[i]|S[i-1]\dots S[1])}_{\to\ \text{seq. non crescente}\ H_\infty(S)}$$

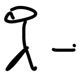
$$\lim_{m\to\infty} \longrightarrow H_\infty(S)$$

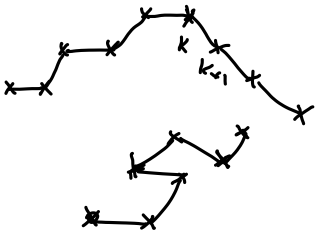
$$\lim_{m\to\infty} \frac{H(S[1]\dots S[m])}{m} = H_\infty(S) = \lim_{n\to\infty} H(S[n]|S[n-1]\dots S[1])$$

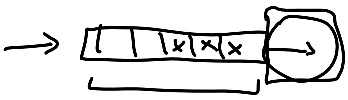
Lezione 5

lunedì 19 ottobre 2020 10.59

PROCESSI DI MARKOV

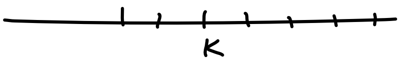






$S(k+1) = f[S(k), u(k)]$:

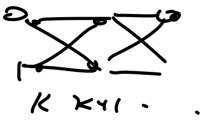
$\{S[K]\}$

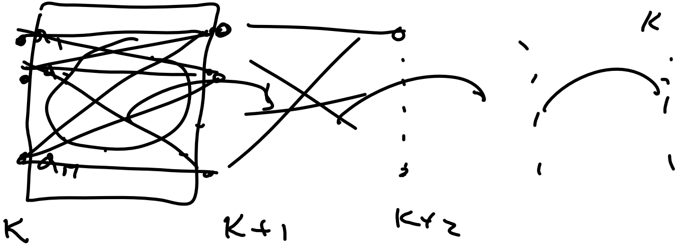


$$P_r\{S[K]=s_K \mid S[K-1]=s_{K-1}, S[K-2]=s_{K-2}, S[K-3]=s_{K-3}, \ldots\ldots\}$$

$$= P_r\{S[K]=s_K \mid S[K-1]=s_{K-1}\} \Rightarrow$$ Processo di Markov.

$S[K] \in \Omega = \{a_1, \ldots, a_M\}$





TRELLIS (TRALICCIO)

$$P_r\{S[K+1] \mid S[K]\} = P_r\{S[K+2] \mid S[K+1]\} = P_r\{S[K+m] \mid S[K+m-1]\}$$

STAZIONARIA O OMOGENEA $\forall m$

$$\begin{array}{c|ccc} & a_1 & a_2 & a_M \\ \hline a_1 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot \\ a_M & \cdot & \cdot & \cdot \end{array} = \begin{bmatrix} P\{S[K+1]=a_1 \mid S[K]=a_1\} & P\{S[K+1]=a_2 \mid S[K]=a_1\} & \ldots P_1\{S[K+1]=a_M / S[K]=a_1\} \\ P\{S[K+1]=a_1 \mid S[K]=a_2\} & P\{S[K+1]=a_2 \mid S[K]=a_2\} & \ldots\ldots \\ & X[K+1] & \end{bmatrix}$$

$$X(K) \quad \begin{bmatrix} p(1|1) & p(2|1) & \ldots p(M|1) \\ p(1|2) & p(2|2) & p(M|2) \end{bmatrix} \rightarrow 1$$

row-sum . 1.

$x(k)$

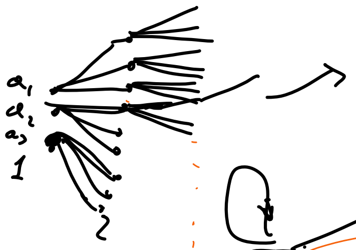
$$\underline{P} = \begin{bmatrix} p(1|1) & p(2|2) & p(M|2) \\ p(1|M) & p(2|M) & p(M|M) \end{bmatrix}$$ row-stochastic

$1 \quad 2 \quad 3 \; . \; . \; . \; - - - \; K \; . \; - - \; . \; .$

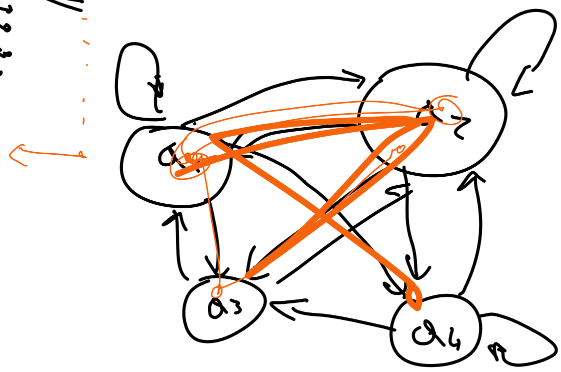
$$P\{X[1]\} = \underline{P} = \begin{bmatrix} P_r\{X[1]=a_1\} \\ P_r\{X[1]=a_2\} \\ \vdots \\ P_r\{X[1]=a_M\} \end{bmatrix}$$

POSSIBILI RAPPRESENTAZIONI

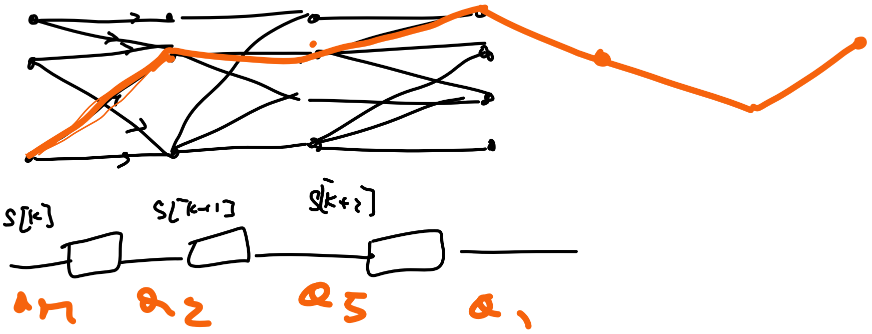
ALBERO



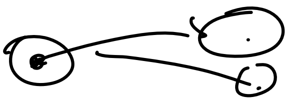
crescita esponenziale.



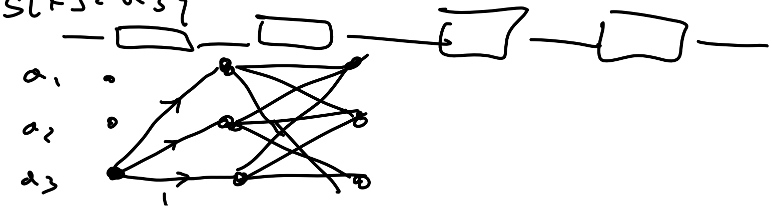
grafo di stato.

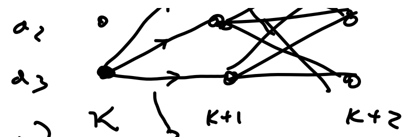


CATENA ERGODICA o IRRIDUCIBILE



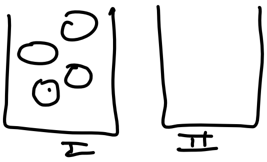
$P_r\{s[k+2]=\cdot \mid s[k]=a_3\}$





$$P_r\{S[K+2] \mid S[K]\} = P^{T^2} p(S[K])$$

$$P_r\{S[K+m] \mid S[K]\} = \left(\underline{\underline{P}}^T\right)^K p(S[K])$$

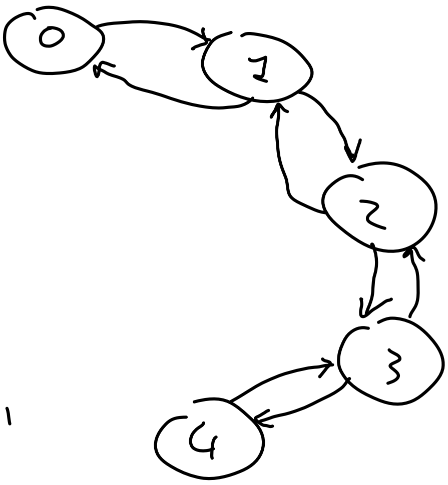


4 palle 2 Urne

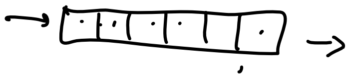
ad ogni passo prendo una palla a caso e la passo nell'altra urna

S[K] = # palle nell'urna I

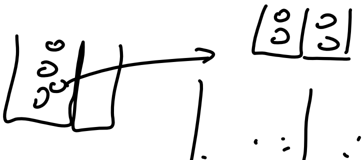
S[K+1] \ S[K]	0	1	2	3	4
0	0	1	0	0	0
1	1/4	0	3/4	0	0
2	0	1/2	0	1/2	0
3	0	0	3/4	0	1/4
4	0	0	0	1	0

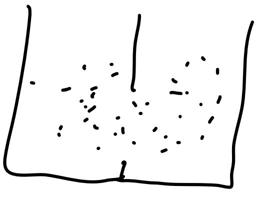


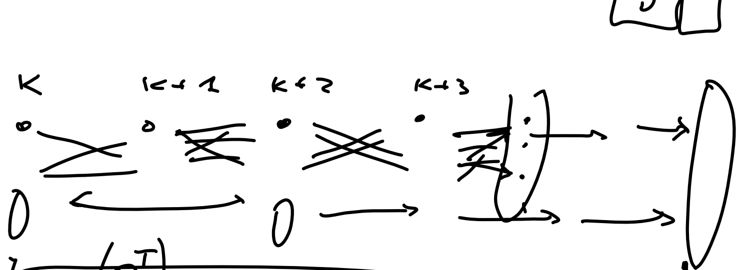
BUFFER



S[K] = {0, 1, 2, 3, ..., m}







$(P^T)^K \quad K \to \infty$

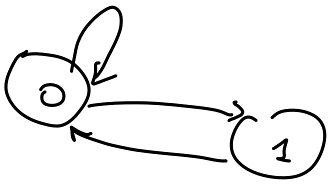
$$p^K \longrightarrow \begin{bmatrix} w^T \\ w^T \\ w^T \end{bmatrix} \qquad p^K = (P^T)^K p^K$$

$P \longrightarrow \begin{bmatrix} \underline{w}^T \\ \underline{w}^T \\ \underline{w}^T \end{bmatrix}$ $\qquad P = (1 \quad) P$

$$\boxed{p[\infty] = P^T \, p[\infty]}$$

$\| \qquad \square \qquad \pi$

$$\begin{cases} \underline{\pi} = \underline{P}^T \underline{\pi} \\ \pi \text{ stocastico.} \end{cases} \qquad \pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_m \end{bmatrix} \quad \sum_{i=1}^{m} \pi_i = 1$$



$$P = \begin{bmatrix} 0.1 & 0.9 \\ 1 & 0 \end{bmatrix}$$

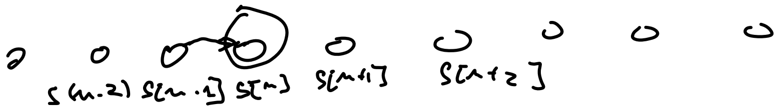
001001010

$\underset{\cdot}{0} \longrightarrow \longrightarrow \longrightarrow 1$

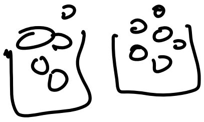
$$\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = (P^T)^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Lezione 6

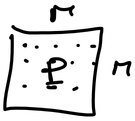
venerdì 23 ottobre 2020 11.02



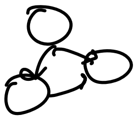
$$p(S[n] \mid S[n-1]\, S[n-2] \ldots S[1])$$
$$= p(S[n] \mid S[n-1])$$



$p(S[1])$ $P(S[n] \mid S[n-1])$



$$p(S[n]) = \left(P^T\right)^{n-1} p(S(1))$$



Ergodica

Regolare.

$$p(\infty) = \lim_{k\to\infty} \left(P^T\right)^k$$

$$p(\infty) = P^T p(\infty)$$

$$H_\infty(S) = \lim_{n\to\infty} H(S[n] \mid S[n-1]\, S[n-2] \ldots S[1])$$

$$\lim_{k\to\infty} \frac{H(S^k)}{k} = H_\infty(S) \quad \text{se } \{s[n]\} \text{ staz.}$$

CATENA DI MARKOV

$$H_\infty(S) = \lim_{n\to\infty} H(S[n] \mid S[n-1])$$
$$= H(S[n] \mid S[n-1]) \quad \forall n > n_0$$

$$H(X|Y) = \sum_{(x,y)} \overset{p(x|y)p(y)}{\overline{p(x,y)}} \log_2 \frac{1}{p(x|y)} = E_{x,y}\left[\log_2 \frac{1}{p(x|y)}\right]$$

$$S[n] \in \{a \ldots a_M\} = \mathcal{X}^2$$

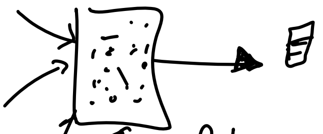
$$H(S[n] \mid S[n-1]) = \sum_{(\xi,\eta)\in\Omega^2} p(\xi|\eta)\, \underbrace{p(\eta)}_{P(S[n-1]) = p(S[n]) = p(\infty)} \log_2 \frac{1}{\underbrace{p(\xi|\eta)}_{\underline{\underline{P}}}}$$

$=$

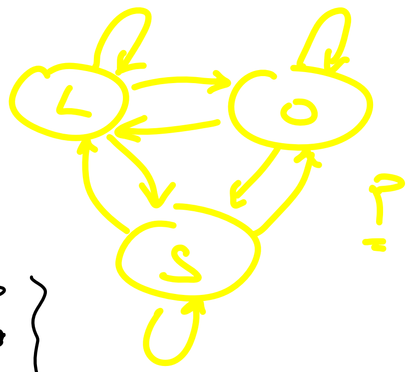
$$H_\infty(S) = -\underline{p}^T[\infty] \left(\underline{\underline{P}} \odot \log_2 \underline{\underline{P}}\right) \underline{e} \quad \text{bit}$$

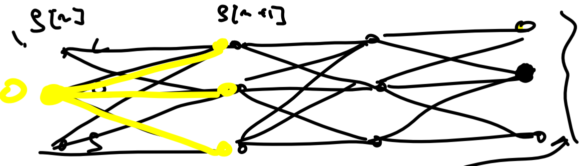
$$\underline{e} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

CATENE DI MARKOV NASCOSTE



$S[n]$
- L libero → servizio
- O occupato → ...
- S spento → ....

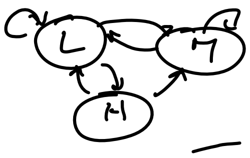




$$P \overset{?}{=} P_S[\infty] = (P^T)^3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

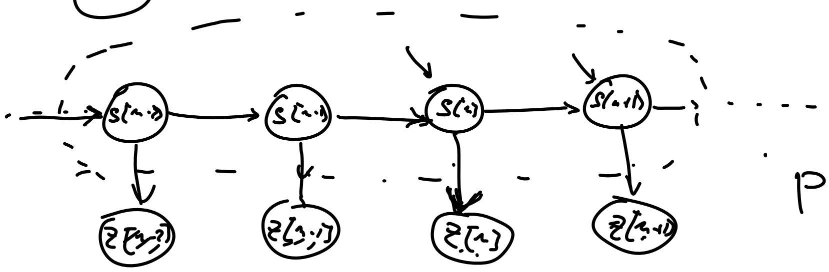
Temperatura del Processore

- L
- M
- H



$$[\ ] = (P^T)^{10} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

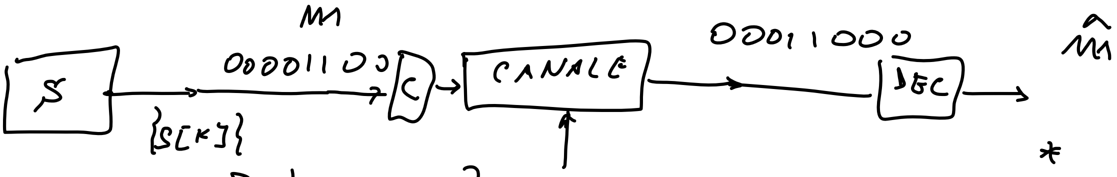
HMM
HIDDEN
MARKOV
MODEL



$$Z[n] \in \mathcal{Z} = \{b_1 \dots b_M\}$$

$$P(Z/S) = \left[P_r\{Z = b_i / S = a_j\} \right]$$

$$\begin{cases} S[n+1] = f(S[n], w[n]) \\ Z[n] = g(S[n]) \end{cases}$$

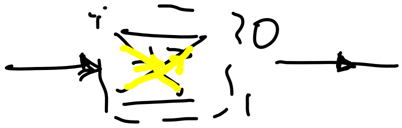


$S[K] \in \mathcal{A} = \{a_1, \dots, a_M\}$

RIDONDANZA

0

1



Trasmessa	0	0	1	0	1
Ricevuta	0	1	0	0	1

CODICE A RIPETIZIONE

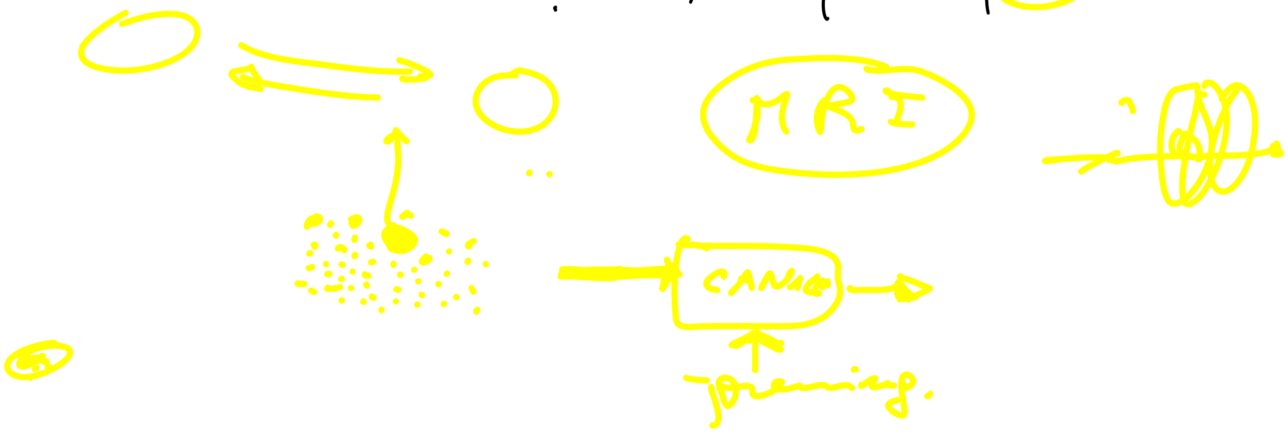
00:00:11:00:11

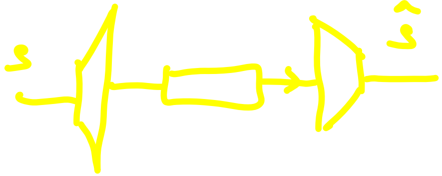
~~Ricevuta~~ 1 0:01:11:10:11

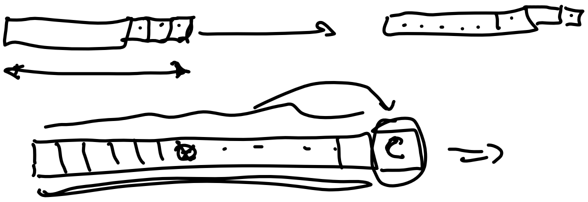
000:000 111 1 000 111

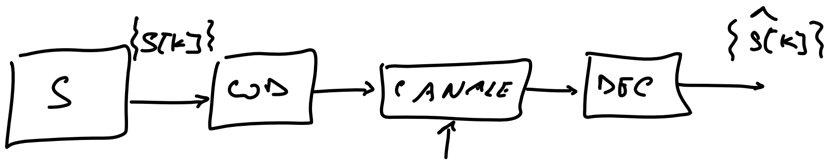
RIPETIZ

Ricevuto	1 0	0 1	1 1	1 0	1 1
Ric.	000 010	000 000	111 011	000 000	111 100
	0	0	1	0	0

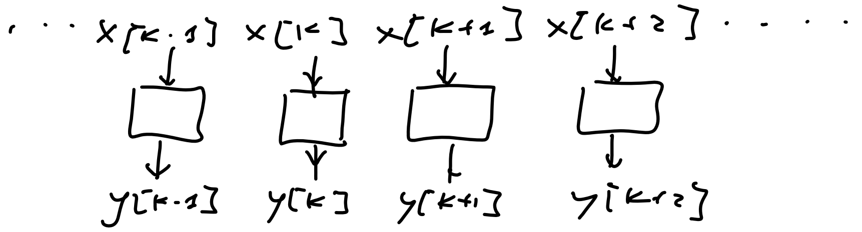








Semplificazione



Esempio

$$\begin{array}{cccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}$$

$$\mathcal{X}=\{0,1\} \qquad \begin{array}{cccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & ? & 0 & 0 & ? & 1 & 1 & 0 \end{array}$$

$$\mathcal{Y}=\{0,1,*\}$$

TEST COVID:

$$\mathcal{X}=\{c,\bar{c}\} \qquad \begin{array}{cccccccc} c & \bar{c} & c & \bar{c} & \bar{c} & c & \bar{c} & \bar{c} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \bar{c} & \bar{c} & c & c & \bar{c} & * & c & \bar{c} \end{array}$$

$$\mathcal{Y}=\{c,\bar{c},*\}$$

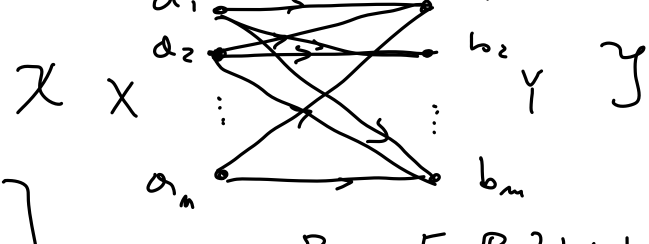
Falso negativo   falso positivo ↗

Testo

$$\mathcal{X}=\{\underbrace{\_\,\_\,\_\,\_\,\_\,\_}_{128 \text{ caratteri}}\} \qquad \begin{array}{ccccccccccc} A & B & R & A & C & A & D & A & B & R & A \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S & B & T & A & C & Z & D & Q & H & R & A \end{array}$$

$$\mathcal{Y}=\{\underbrace{\_\,\_\,\_\,\_\,\_\,\_}_{128 \text{ caratteri}}\}$$

CANALE

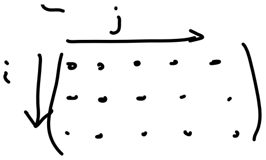


$$\begin{bmatrix} P_r\{X=a_1\} \\ P_r\{X=a_2\} \\ \vdots \\ P_r\{X=a_m\} \end{bmatrix}$$

$$\underline{\underline{P_c}} = \left[P_r\{Y=b_j \mid X=a_i\} \right]_{\substack{i=1:m \\ j=1:m}}$$

$$\overset{j \rightarrow}{} \qquad i \downarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\begin{bmatrix} P_r\{X=a_2\} \\ \vdots \\ P_r\{X=a_n\} \end{bmatrix}$$



$j=1:m$

$$\underline{\underline{P}}_c = n \begin{bmatrix} P_r\{Y=b_1|X=a_1\} & P_r\{Y=b_2|X=a_1\} & \dots & P_r\{Y=b_m|X=a_1\} \\ P_r\{Y=b_1|X=a_2\} & P_r\{Y=b_2|X=a_2\} & \dots & P_r\{Y=b_m|X=a_2\} \\ \vdots & & & \\ P_r\{Y=b_1|X=a_n\} & - & - - - & P_r\{Y=b_m|X=a_n\} \end{bmatrix}$$

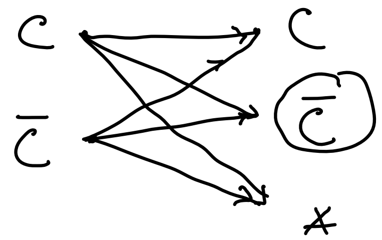
MATRICE DI CANALE

$$\xleftrightarrow[m]{}$$

COVID

$\mathcal{X}=\{c,\bar{c}\}$

$\mathcal{Y}=\{c,\bar{c},*\}$



$$\underline{\underline{P}}_c = \begin{bmatrix} p(c|c) & p(\bar{c}|c) & p(*|c) \\ p(c|\bar{c}) & p(\bar{c}|\bar{c}) & p(*|\bar{c}) \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.08 & 0.02 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}$$

$P_r\{Y=c, X=c\}$?

$P_r\{Y=\bar{c}, X=\bar{c}\}$?

$P_r\{Y=*\} =$?

$P_r\{Y=c ; X=\bar{c}\}$?

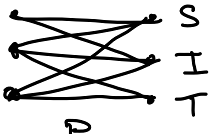
$p(Y=c, X=c) = p(Y=c|X=c)\, P\{X=c$

:

canale.

$p(S)=0.4$
$p(I)=0.4$
$p(T)=0.2$

S	STRESS
I	IPERTENSIONE
T	TUMORE

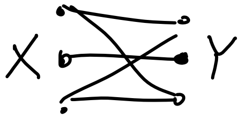


$\underline{\underline{P}}_c$

	S	I	T
S	0.5	0.5	0
I	0.19	0.8	0.01
T	0.2	0.2	0.4

$\mathcal{X}=\{a_1 \dots a_n\}$ $\mathcal{Y}=\{b_1 \dots b_m\}$

$$\underline{\Pi}_x = \begin{bmatrix} P_r\{X=a_1\} \\ \vdots \\ P_r\{X=a_n\} \end{bmatrix}$$



$$\underline{\underline{P}}_c = \left[P_r\{Y=b_j|X=a_i\} \right]^{j=1..m}_{i..}$$

$$\left[P_r\{X=a_n\} \right] \qquad \underline{\underline{P}}_c = \left[P_r\{Y=b_j \mid X=a_i\} \right]_{i=1..n}^{j=1..m}$$

$$P_r\{Y=b_j\} = \sum_{i=1}^{n} P_r\{Y=b_j \mid X=a_i\} P_r\{X=a_i\}$$

$$= \sum_{i=1}^{n} P_r\{Y=b_j, X=a_i\}$$

$$\underline{\underline{\pi}}_y = \begin{bmatrix} P_r\{Y=b_1\} \\ P_r\{Y=b_2\} \\ \vdots \\ P_r\{Y=b_m\} \end{bmatrix} = \underline{P_c}^T \underline{\pi}_x \qquad \underline{\pi}_y = \underline{\underline{P}}_c^T \underline{\pi}_x$$

$$P_r\{X=a_i \mid Y=b_j\} = \frac{P_r\{X=a_i, Y=b_j\}}{P_r\{Y=b_j\}}$$

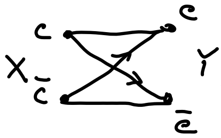
PROB. A POSTERIORI

$$= \frac{P(Y=b_j \mid X=a_i\} P_r\{X=a_i\}}{\sum_{\ell=1}^{n} p(Y=b_j \mid X=a_\ell\} P_r\{X=a_\ell\}}$$

T. DI BAYES

ESEMPIO

$$\pi_X = \begin{bmatrix} P_r\{X=c\} \\ P_r\{X=\bar{c}\} \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$$



$$\underline{\underline{P}}_c = \begin{bmatrix} P_r\{Y=c \mid X=c\} & P_r\{Y=\bar{c} \mid X=c\} \\ P_r\{Y=c \mid X=\bar{c}\} & P_r\{Y=\bar{c} \mid X=\bar{c}\} \end{bmatrix}$$

$$\pi_y = \begin{bmatrix} P_r\{Y=c\} \\ P_r\{Y=\bar{c}\} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.35 & 0.65 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.455 \\ 0.545 \end{bmatrix} \left(= \begin{bmatrix} 0.7 & 0.3 \\ 0.35 & 0.65 \end{bmatrix} \right)$$

$$P_r\{X=c \mid Y=\bar{c}\} = \frac{P_r\{Y=\bar{c} \mid X=c\} P(X=c)}{P(Y=\bar{c})} = \frac{0.3 \cdot 0.3}{0.545} = 0.16$$

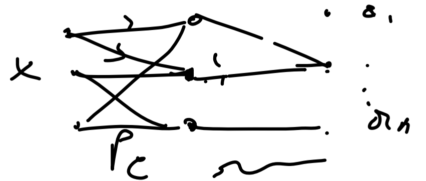
$$P_r\{X=\bar{c} \mid Y=\bar{c}\} = 0.84$$

$$\underline{\underline{P}}_p = \begin{bmatrix} P(X=a_1 \mid Y=b_1) & \\ & \text{prob. a posteriori} \end{bmatrix}$$

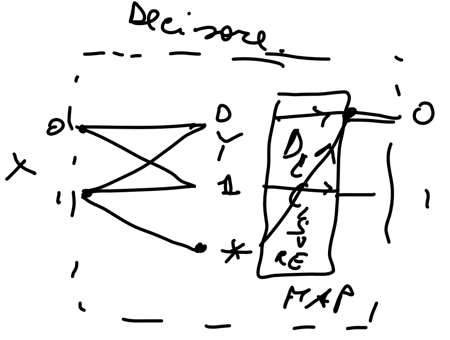
MASSIMA PROB. A POSTERIORI

$$\hat{a}_\ell = \underset{i=1..m}{\operatorname{argmax}} P_r\{X=a_i \mid Y=b_j\}$$

MAP



Esempio



$\mathcal{X} = \{0,1\}$

$\mathcal{Y} = \{0,1,K\}$

$$\Pi_X = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

0 0 0 0 $\underset{q}{0}$ 1 0 0 0 1 1 0 0 0 0

$$P_C = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$$

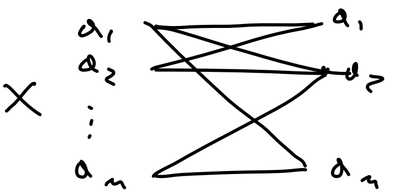
$$P_P = \begin{bmatrix} P(X=0|Y=0) & P(X=0|Y=1) & P(X=0|Y=K) \\ P(X=1|Y=0) & P(X=1|Y=1) & P(X=1|Y=K) \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{0.9697} & 0.3333 & \boxed{0.8000} \\ 0.0303 & \boxed{0.6667} & 0.200 \end{bmatrix}$$

PROB. DI ERRORE

$m = n$

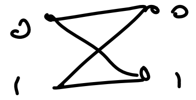
$\mathcal{X} = \{a_1 .. a_n\}$ $\mathcal{Y} = \{a .... a_n\}$



0 1 1 0 0 1 0

↓ ↓ ↓ ↓ ↓ ↓ 2

0 1 0 0 1 0 0



A B R A C A D

A N R Z C A D

$$P_r\{\text{errore}\} = P_r\{X \neq Y\} = \sum_{\substack{i \neq j \\ i,j=1}}^{n} P_r\{x=a_i, y=a_j\}$$

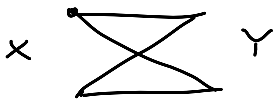
$$= 1 - P_r\{\underset{\text{corretta ric.}}{\text{non errore}}\} = 1 - \sum_{i=1}^{n} P_r\{x=a_i, y=a_i\}$$

$$= 1 - \sum_{i=1}^{m} \Pr\{y = a_i \mid x = a_i\} \Pr\{X = a_i\}$$

$$= 1 - (\text{diag}\, \underline{P}_c)^T \underline{\pi}_x$$

Esempio

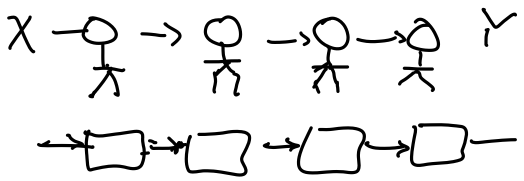
$$\pi_x = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

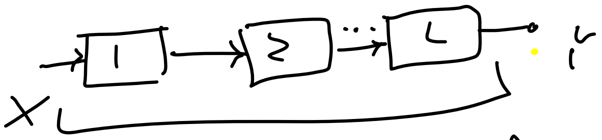


$$P_c = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

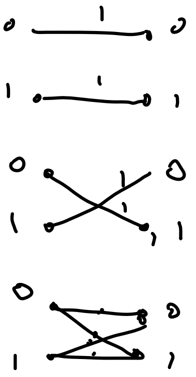
$$P_e = 1 - [0.8 \quad 0.9] \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = 0.18$$

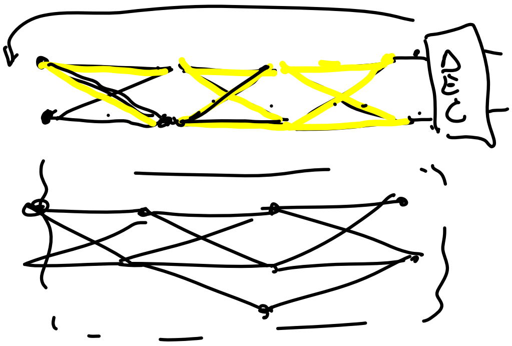
CANALI IN CASCATA





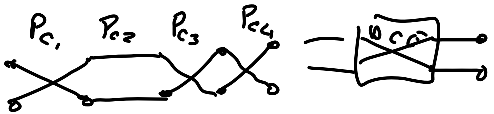
$$P_{ct} = P_{c_1} P_{c_2} \dots P_{c_L}$$



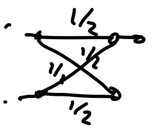


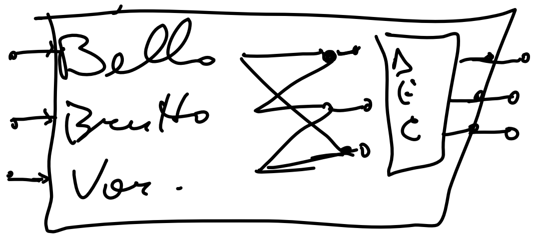
$$\pi_y = P_{ct}^T \pi_x =$$

$$= P_{c_L}^T P_{c_{L-1}}^T \dots P_{c_1}^T \pi_x$$

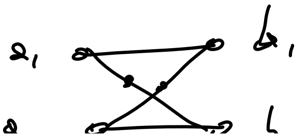


$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

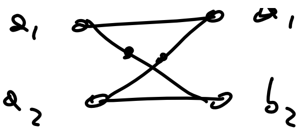




CANALE BINARIO



$$\begin{pmatrix} \cdot & & \cdot \\ & \underline{P}_c & \\ \cdot & & \cdot \end{pmatrix}$$



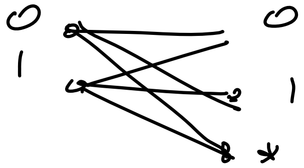
$$\begin{pmatrix} \cdot & & \cdot \\ & P_c & \\ \cdot & & \cdot \end{pmatrix}$$

CANALE BINARIO SIMMETRICO

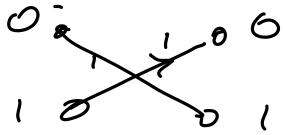
BSC

$$\underline{P}_c = \begin{pmatrix} 1-p_e & p_e \\ p_e & 1-p_e \end{pmatrix}$$

CANALE CON CANCELLAZIONE



CANALE SEMPRE IN ERRORE



$$X \rightarrow \boxed{P(Y|X)} \rightarrow Y$$

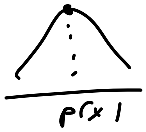
CANALE

$$I(X;Y) = H(X) - H(X|Y)$$

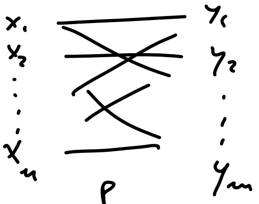
$$p(x),\ p(y|x)$$

$$C = \max_{p(x)} I(x;y)$$

CAPACITA' DI CANALE [bit]



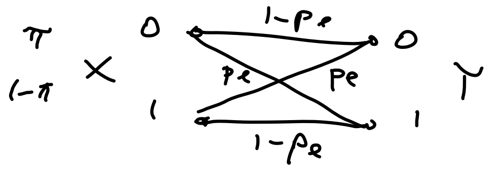
$C \geq 0$



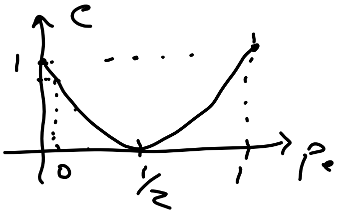
$\underset{=}{P_c}$

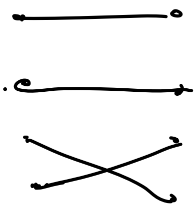
$$C \leq \log_2 n \qquad C \leq \log_2 m$$

Esempio



$$C = 1 - H(p_e)$$



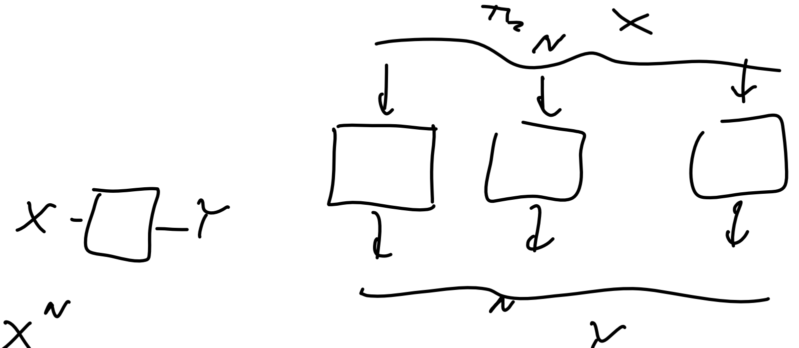


$p_e = 0.1$

$$\boxed{C = 0{,}53}$$

$\underset{=}{P_c},\ \underset{.}{\pi_x} \Longrightarrow C$

ARIMOTO/
BLAUTH

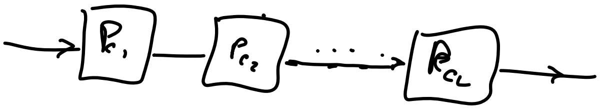


Canale esteso

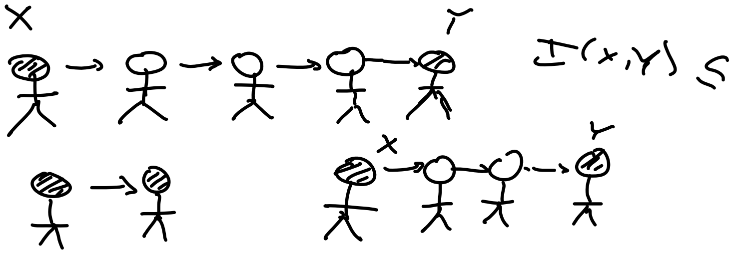
$$P_c^N = \underbrace{P_c \otimes P_c \otimes \cdots \otimes P_c}_{N}$$

$$I(X^N;Y^N) = N I(X;Y)$$

$$\boxed{C^N = N C}$$



TEOREMA DEL DATA PROCESSING
TRATTAMENTO DATI

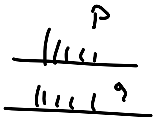


$I(x,Y) \leq$

$$X \xrightarrow{P_{c_1}} Z \xrightarrow{P_{c_2}} Y$$

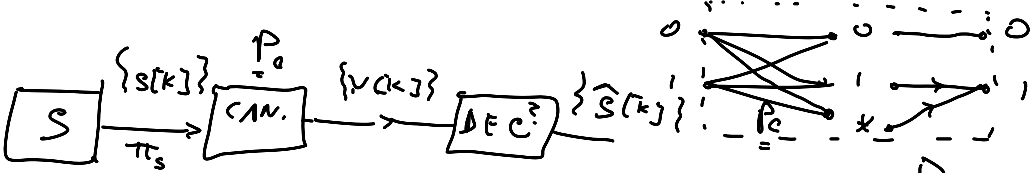
$$I(X;Y) \leq I(X;Z)$$

Divergenza di Kullback-Leibler $p \quad q$



$$D(p\|q) \triangleq \sum p \log \frac{p}{q} \qquad \geq 0$$

$$D(q\|p) \triangleq \sum q \log \frac{q}{p}$$



$$\underset{=}{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$S[k]$	0	1	1	0	0	1	1	0
$Y[k]$	1	0	1	0	*	1	1	*
$\hat{S}[k]$	?	?	?	?	?			

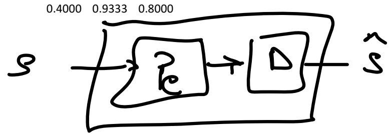
$$\xrightarrow{Y}$$

$$\underset{=}{P_e} = {}^{x}_{1}\begin{bmatrix} p(0|0) & p(1|0) & p(*|0) \\ p(0|1) & p(1|1) & p(*|1) \end{bmatrix} \Rightarrow \underset{=}{P_p} = \begin{bmatrix} P(X=0|Y=0) & p(X=0|Y=1) & P(X=0|Y=*) \\ p(X=1|Y=0) & p(x=1|Y=1) & P(x=1|Y=*) \end{bmatrix}$$

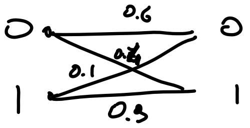
Esempio $\pi_x = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \qquad \underset{=}{P_e} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \end{bmatrix}$

Pp =

```
0.6000   0.0667   0.2000
0.4000   0.9333   0.8000
```



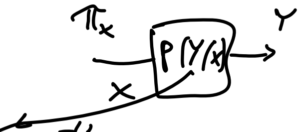
$$P_t = P_c \cdot D = \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}$$



$$P_e = 1 - P_c = 1 - 0.2 \cdot 0.6 - 0.8 \cdot 0.9$$
$$= 1 - \pi_x^T \text{diag}\, P_c = 0.32$$

Decisore (i. di Bayes) MAP

MAXIMUM A POSTERIORI



$p(x|Y) =$

funzione di verosimiglianza likelihood

$$p(X|Y) = \frac{p(Y|X)\,\boxed{p(X)}}{p(y)} \rightarrow \text{uniforme.}$$

MAP $p(x|Y=\bar{y}) \rightarrow$ scelgo $X = \hat{x}$: $p(\hat{x}|Y=\bar{y}) > p(x|Y=\bar{y})$ $\forall x$

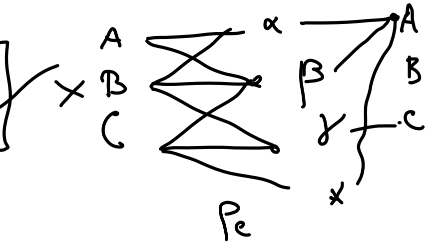
$$p(x|Y=\bar{y}) \propto p(Y=\bar{y}|x)$$

ML (Maximum Likelihood)
Massima verosimiglianza
MV

scelgo $X = \hat{x}$: $p(Y=\bar{y}|\hat{x}) > p(Y=\bar{y}|x)$ $\forall x$

$X \in \mathcal{X} = \{A, B, C\}$ $\quad Y \in \{\alpha, \beta, \gamma, \times\}$

$$P_x = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.3 \end{bmatrix}$$

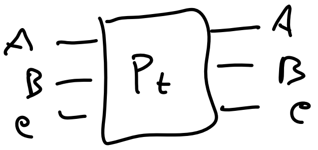


$$P_c = \begin{array}{c|cccc} X \rightarrow Y & & & & \\ \hline & 0.3 & 0.2 & 0.1 & 0.4 \\ & 0.2 & 0.3 & 0.2 & 0.3 \\ & 0.1 & 0.1 & 0.5 & 0.3 \end{array}$$

```
Pp =

    0.6818   0.5263   0.2083   0.5714
    0.1818   0.3158   0.1667   0.1714
    0.1364   0.1579   0.6250   0.2571
```

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



MAP

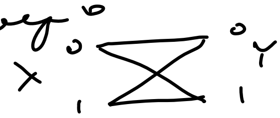
$P_t = P_c D =$

```
Pt =

    0.9000        0   0.1000
    0.8000        0   0.2000
    0.5000        0   0.5000
```

$$P_e = 1 - \pi_x^T \text{diag}(P_c) = 0.4$$

Esempio



$$P_c = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \qquad P_e = 0.1$$

$$\pi_x = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$$

00001 0000 0 ~ 1 1 0 2 1 0 ~

$\pi_x = \begin{bmatrix} \\ 0.05 \end{bmatrix}$

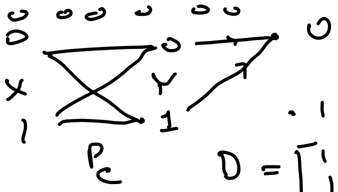
0 0 0 0 1 0 0 0 0 0 ~ 1 1 0 2 1 0 0 0

U 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

$P_p =$

Pp =

0.9942	0.6786
0.0058	0.3214



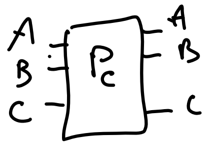
$D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$P_t = P_c D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$P_e = 1 - \pi_x^T diag(P_t) = 1 - (0.95\ 0.01)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.05$;!!

π_x

$H(X)$



$P_e =$ ——— Pratica

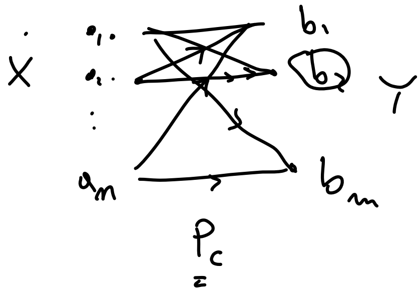
$I(X;Y)$?

C ?

Teoriche della

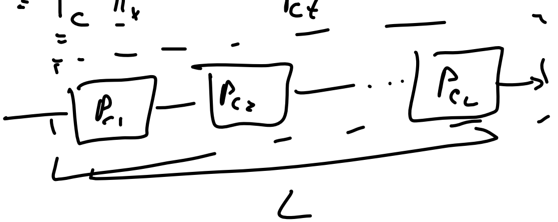
T. I

$x \in \mathcal{X}$



$\underline{\pi}_Y = \underline{\underline{P}}_c^T \underline{\pi}_x$

P_{ct}

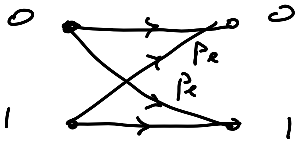


$P_r\{x \mid y = \bar{y}\}$ t. di Bayes.

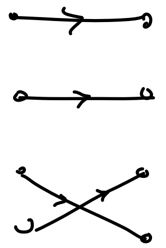
$\underbrace{P_{cL}^T \cdots \underline{\underline{P}}_{c2}^T \underline{\underline{P}}_{c1}^T}\ \underline{\pi}_x$

$P_{ct} = P_{c1} P_{c2} \cdots P_{cL}$

Esempi di canale discreto

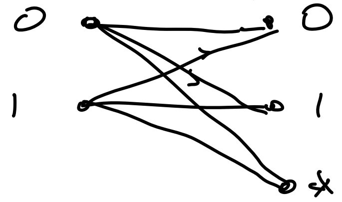


simmetrico



$$\underline{\underline{P}}_e = \begin{bmatrix} p(0|0) & p(1|0) \\ p(0|1) & p(1|1) \end{bmatrix} = \begin{pmatrix} 1-p_e & p_e \\ p_e & 1-p_e \end{pmatrix}$$

canale con cancellazione



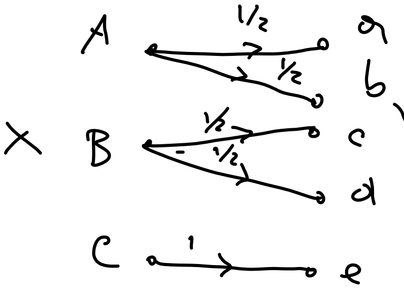
$0\ 1\ 1\ 0\ 0\ 1\ \overset{1}{0}\ 0\ 0\ 1$

$1\ 0\ 0\ *\ *\ 1\ 1\ 0\ 0\ 1$

$$\underline{\underline{P}}_e = \begin{vmatrix} p(0|0) & p(1|0) & p(*|0) \\ p(0|1) & p(1|1) & p(*|1) \end{vmatrix}$$

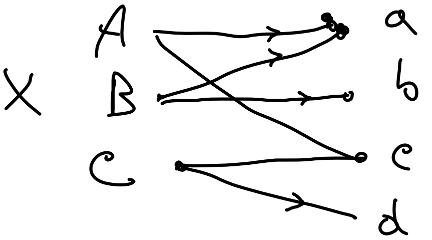
caso simmetrico

$$= \begin{pmatrix} 1-p_e-p_c & p_e & p_c \\ p_e & 1-p_e-p_c & p_c \end{pmatrix}$$

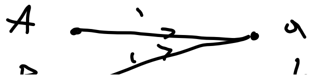


$$\underline{\underline{P}}_c = \begin{array}{c|ccccc} & a & b & c & d & e \\ A & 1/2 & 1/2 & 0 & 0 & 0 \\ B & 0 & 0 & 1/2 & 1/2 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \end{array}$$

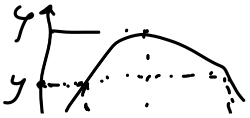
CANALE SENZA RUMORE

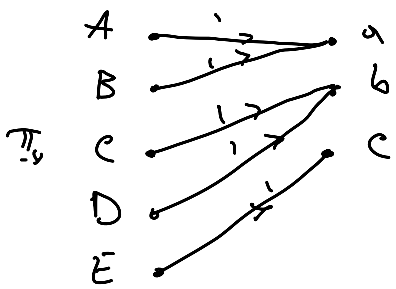


C. DETERMINISTICO

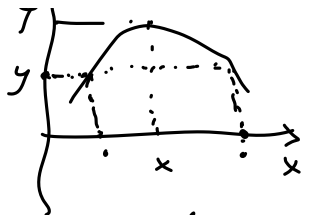


$$\begin{array}{c|ccc} & a & b & c \\ A & 1 & 0 & 0 \\ & 1 & 0 & 0 \end{array}$$

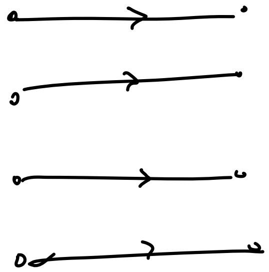


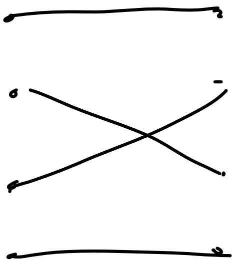


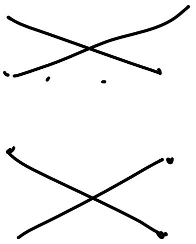
$$P_c = \begin{array}{c}\\ A\\ B\\ C\\ D\\ E\end{array}\begin{array}{ccc} a & b & c \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$



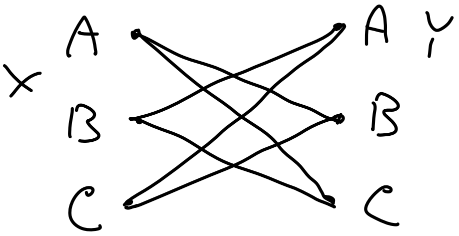
inversione stocastica di una funzione non invertibile (a più valori)





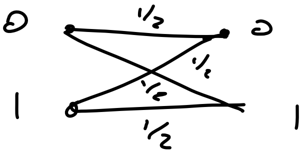


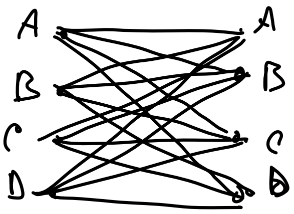
CANALE SEMPRE IN ERRORE



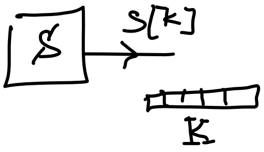
$$P_c = \begin{array}{c|ccc} & \multicolumn{3}{c}{Y} \\ & A & B & C \\ \hline X\ A & 0 & x & x \\ B & x & 0 & x \\ C & x & x & 0 \end{array}$$

CANALE MASSIMAMENTE CONFUSO



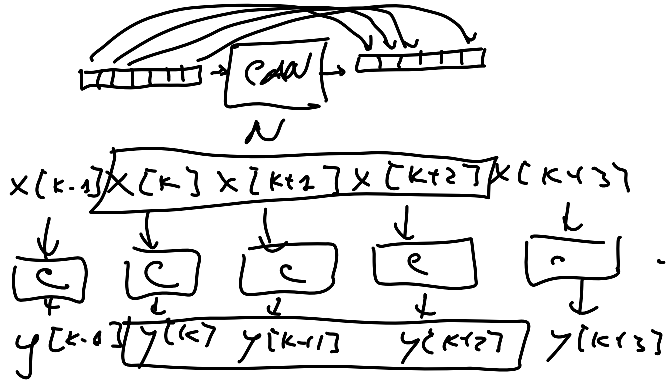


$$P_c = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1/4 & \frac{1}{4} & 1/4 \\ \frac{1}{4} & 1/4 & 1/4 & 1/4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$



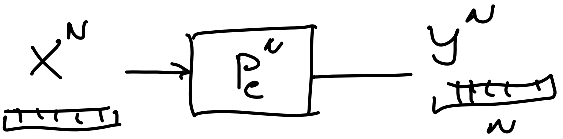
$s \in \mathcal{A}$

$$s^K \in \mathcal{A}^K = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{K}$$



$X^N \in \mathcal{X}^N$ $\qquad$ $y^N \in \mathcal{Y}^N$ $\qquad$ $\mathcal{Y}=\{b_1 \dots b_m\}$

$\mathcal{X}=\{x_1 \dots x_m\}$



$\mathcal{Y}^N=\{\underbrace{\cdot\ -\ -\ -\ -\ -\ \cdot}_{m^N}\}$

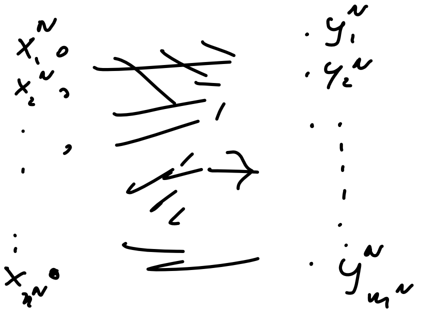
Es.

$\mathcal{X}=\{a\, b\, c\}$ $\qquad 3^2$

$\mathcal{X}^2=\{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

$\mathcal{X}^3=\{aaa, \dots\ \overset{3^3}{-}\ \dots \}$

$\mathcal{X}^N=\{\ -\ -\ \overset{n^N}{\cdot}\ -\ -\ \}$



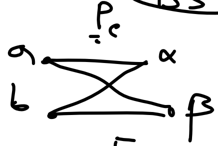
$$\underline{\underline{P}}_c^N = \underbrace{\underline{\underline{P}}_c \otimes \underline{\underline{P}}_c \otimes \dots \otimes \underline{\underline{P}}_c}_{N}$$

$\boxed{BSC}$

Esempio

$\mathcal{X}=\{a, b\}$

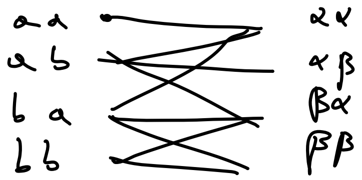
$N=2$



$\mathcal{Y}=\{\alpha, \beta\}$

$$\underline{\underline{P}}_c = \begin{bmatrix} 1-p_e & p_e \\ p_e & 1-p_e \end{bmatrix}$$

$\mathcal{X}^2=\{aa, ab, ba, bb\}$ $\qquad$ $\mathcal{Y}^2=\{\alpha\alpha, \alpha\beta, \beta\alpha, \beta\beta\}$



$$\underline{\underline{P}}_c^2 = \underline{\underline{P}}_c \otimes \underline{\underline{P}}_c = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \otimes \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

$$= \begin{bmatrix} (1-p)\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} & p\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \\ p\begin{bmatrix} 1-p & p \end{bmatrix} & (1-p)\begin{bmatrix} 1-p & p \end{bmatrix} \end{bmatrix}$$

$$\left[p \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \quad (1-p) \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \right]$$

$$= \begin{bmatrix} (1-p)(1-p) & (1-p)p & p(1-p) & pp \\ (1-p)p & (1-p)(1-p) & pp & p(1-p) \\ p(1-p) & pp & (1-p)(1-p) & (1-p)p \\ pp & p(1-p) & (1-p)p & (1-p)(1-p) \end{bmatrix}$$

$p(x)$

$X \longrightarrow \boxed{p(y|x)} \longrightarrow Y$

$p(x|Y)$ prob. a posteriori

$p(x)$ prob a priori.

incertezza su X dato Y

$$H(X|Y) \triangleq \sum_{(x,y)\in \mathcal{X}\times\mathcal{Y}} p(x,y) \lg_2 \frac{1}{p(x|y)} =$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(y|x)p(x)}{\sum_\xi p(y|\xi)p_x(\xi)}$$

AMBIGUITA'
DEL CANALE
bit.

$$= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x|y)p(y) \lg_2 \frac{1}{p(x|y)}$$

$$= \sum_{(x,y)} p(y|x)p(x) \lg_2 \frac{\sum_\xi p(y|\xi)p_x(\xi)}{p(y|x)p(x)}$$

MUTUA INFORMAZIONE

$X \longrightarrow \boxed{p(Y|x)} \longrightarrow Y$

$H(x)$

$$\boxed{I(X;Y) = H(X) - H(X|Y)}$$

$$I(X;Y) = I(Y;X) = H(Y) - H(Y|X)$$

$$I(X;Y) = H(X) - H(X|Y) \geq 0$$

$$= \sum_x p(x) \lg_2 \frac{1}{p(x)} - \sum_{xy} p(x,y) \lg_2 \frac{1}{p(x|y)}$$

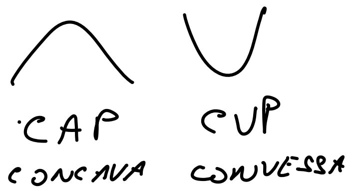
$$p(x) = \sum_y p(x,y)$$

$$= \sum_{xy} p(x,y) \lg \frac{1}{p(x)} - \sum_{xy} p(x,y) \lg_2 \frac{1}{p(x|y)}$$

$$= \sum p(x,y) \lg \frac{p(x|y)}{}$$

$$= \sum_{xy} p(x,y) \log_2 \frac{p(x|y)}{p(x)}$$

$$= \sum_{xy} p(x,y) \log_2 \frac{p(x,y)}{p(y)\,p(x)}$$



$I(X;Y) \geq 0$ $\qquad I(X;Y) = I(Y;X)$

$I(X;Y)$ convessa $\cap$ rispetto a $p(x)$

$I(X;Y)$ ... $\cup$ rispetto a $p(y|x)$

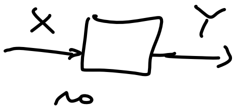
X, Y indipendenti.

$$I(X;Y) = H(X) - \underset{H(X)}{\underset{\shortparallel}{H(X|Y)}} = 0$$

$p(x,y) = p(x)p(y)$
$p(y|x) = p(y)$
$p(x|y) = p(x)$

---

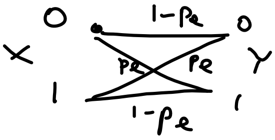
canale senza ambiguità



$H(X|Y) = 0$

$I(X;Y) = H(X)$

$$\Pi_x = \begin{bmatrix} p \\ 1-p \end{bmatrix}$$



$$I(X;Y) = \underset{1}{H(X)} - \underset{\leq 1}{\underline{H(X|Y)}}$$

$$H(X|Y) = \sum_x \sum_y p(x,y) \log_2 \frac{1}{p(x|y)}$$

$$H(X) = \sum_x p(x) \log_2 \frac{1}{p(x)} = p \log \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \leq \log_2 2 = 1$$

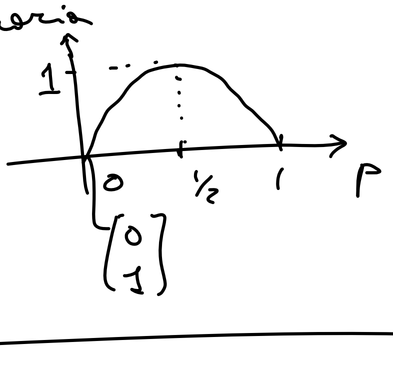
p
$1-p$

$$\log \frac{1}{p_x} \quad [\;\cdot\;\;\cdot\;] \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \quad \boxed{-p_x^T \cdot \log_2 p_x}$$

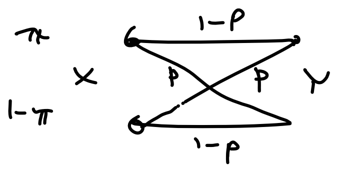
$H(X)$ sorgente binaria

$$p(x) = \begin{bmatrix} p \\ 1-p \end{bmatrix}$$

$$H(x) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}$$



1-p

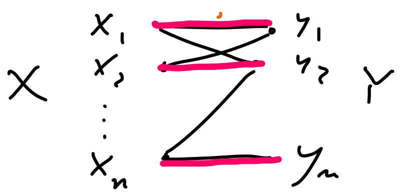


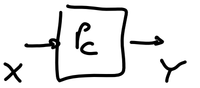
$$
\begin{aligned}
I(X;Y) &= ((1-p)\pi + p(1-\pi)) \log_2 \frac{1}{(1-p)\pi + p(1-\pi)} \\
&\quad + (p\pi + (1-p)(1-\pi)) \log_2 \frac{1}{p\pi + (1-p)(1-\pi)}
\end{aligned}
$$

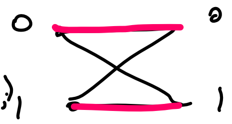
Lezione 10

venerdì 6 novembre 2020 11.03

$$P(e) \longleftrightarrow H(X),\ H(X|Y),\ I(X;Y),\ C$$





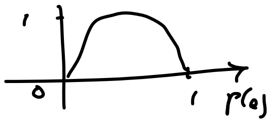


$$P(e) = \Pr\{Y \neq X\} = \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{m} p(x_i,y_j)$$

$$= 1 - \Pr\{Y = X\} = 1 - \sum_{i=1}^{m} P(x_i,y_i)$$

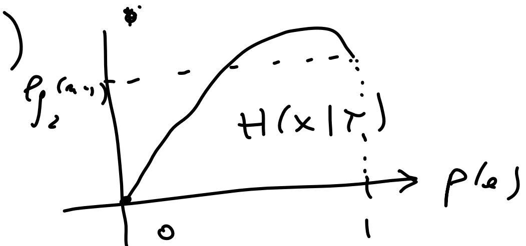
DIS. DI FANO:

$$H(X|Y) \le \boxed{H(e) + P(e)\log_2(n-1)}$$



$$H(e) = -p(e)\log_2 p(e) - (1-p(e))\log_2(1-p(e))$$

$$F(p(e))$$

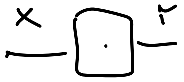


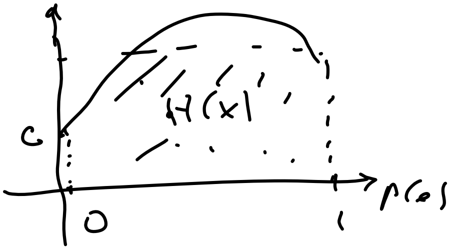
$$I(X;Y) = H(X) - H(X|Y)$$

$$H(X) - I(X;Y) \le H(p(e)) + p(e)\log_2(n-1)$$

$$H(X) \le \underset{\le C}{I(X;Y)} + H(p(e)) + p(e)\log_2(n-1)$$

$$H(X) \le \underbrace{C + H(p(e)) + p(e)\log_2(n-1)}$$



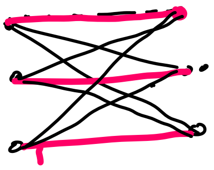


$p(e) = 0$

$$\boxed{H(X) \le C}$$!!!

!!!

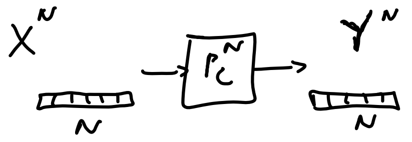
CANALE UNIFORME



$$P_c = \begin{pmatrix} 1-p_e & \frac{p_e}{2} & \frac{p_e}{2} \\ \frac{p_e}{2} & 1-p_e & \frac{p_e}{2} \\ \frac{p_e}{2} & \frac{p_e}{2} & 1-p_e \end{pmatrix}$$

$$\begin{pmatrix} 1-p_e & \frac{p_e}{n-1} & \cdots & \frac{p_e}{n-1} \\ \frac{p_e}{n-1} & 1-p_e & & \vdots \\ & & \ddots & \\ \vdots & \cdots & & 1-p_e \end{pmatrix}$$

$$H(X|Y) \le \underbrace{H(e)} + \underbrace{P(e)\log_2(n-1)}$$



$$P(e_w) = P_r\{X^N \neq Y^N\}$$

DIS. DI FANO

$$H(X^N|Y^N) \le H(e_w) + P(e_w)\log_2(n^N-1)$$

$$C^N = \max_{\pi(X^N)} I(X^N;Y^N) = NC$$

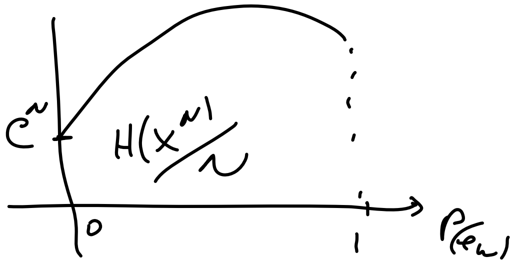
$$H(X^N) \le C^N + H(e_w) + P(e_w)\log_2(n^N-1)$$

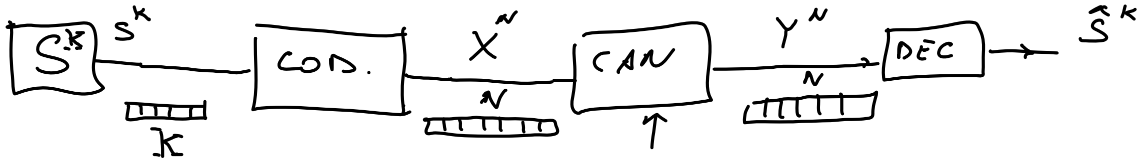
$$H(X^N) \le NC + H(e_w) + P(e_w)\log_2(n^N-1)$$

$$\frac{H(X^N)}{N} \le C + \frac{H(e_w)}{N} + \frac{P(e_w)}{N}\log_2(n^N-1)$$

$$P(e_w) = 0$$

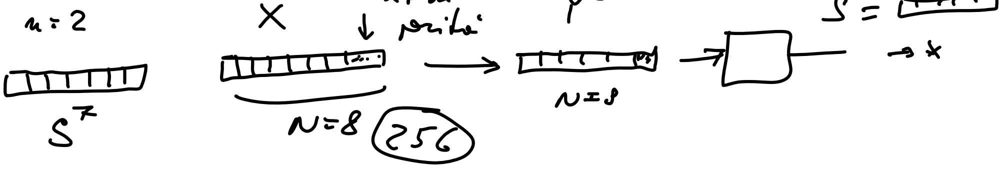
$$\boxed{\frac{H(X^N)}{N} < C}$$





Esempi:

Controllo di parità (1 bit) K = 7 n = 2



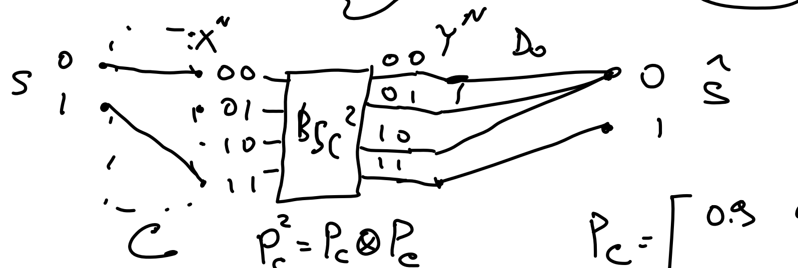
Codice a ripetizione (binario)

~~X~~ , S: 0 1 1 0 0 . . .

$\{X^N\}$: 00 11 11 00 00

$\{Y^N\}$: 10 1 1 0

Y^N	1 0	1 1	0 1	0 0	0 0
S	0	1	1	0	0



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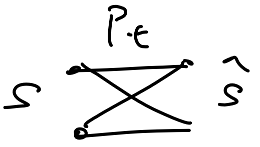


$C P_c^2 D_0$

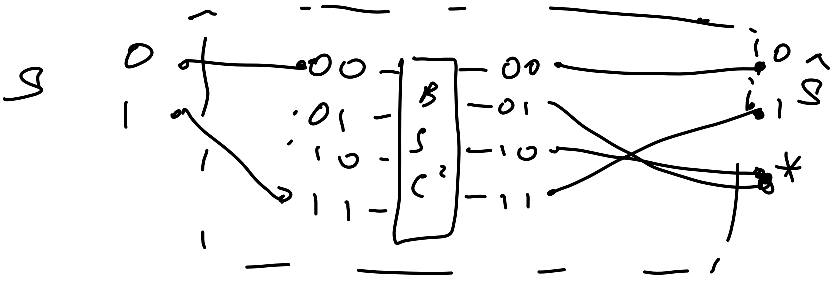
$\underline{P}_c^2 = \underline{P}_c \otimes \underline{P}_c$

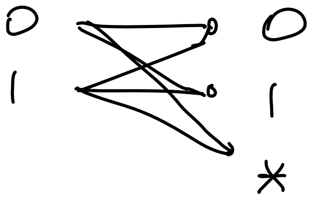
$$\underline{P}_c = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

$$\underline{P}_c^2 = P_c \otimes P_c = \begin{bmatrix} 0.9\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix} & 0.1\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix} \\ 0.1\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix} & 0.9\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix} \end{bmatrix}$$

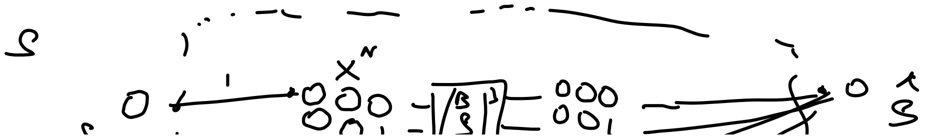
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

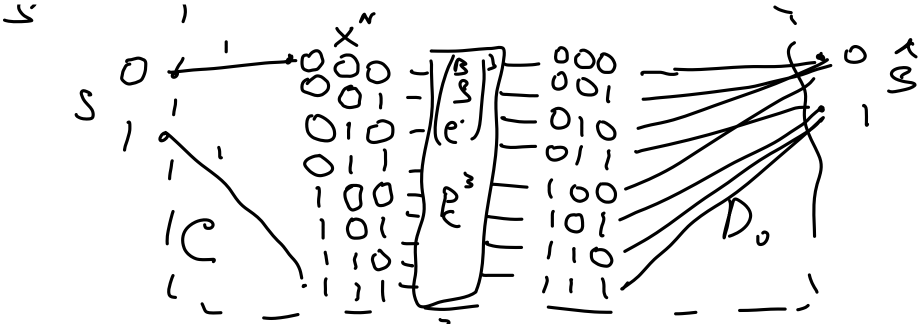
$$\underbrace{C P_c^2} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

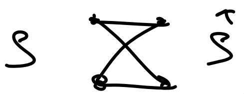




S	0	1	0	0	1
X^N	000	111	000	000	111
Y^N	010	110	100	000	101



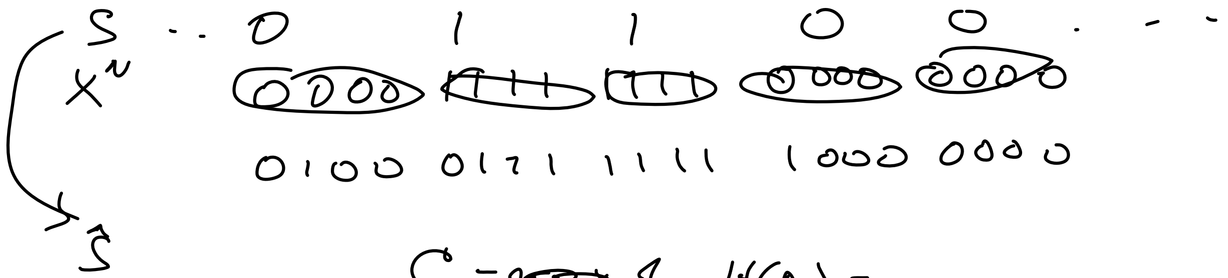




$P_E = C P_c^3 D_0$

$P_c^3 = P_c \otimes P_c \otimes P_c$

$C P_c^3 = (\ldots)$

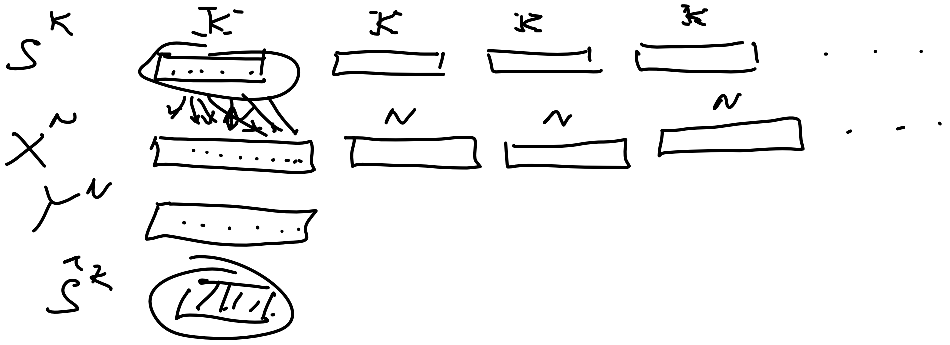


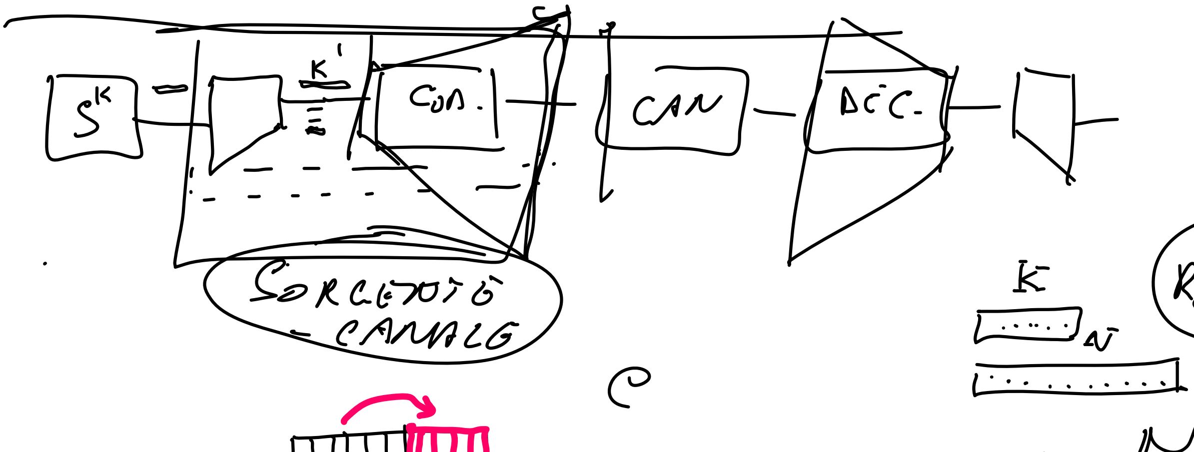
$C = 1 - H(p_e) =$

$R_c = \boxed{\frac{K}{N} < C}$

$C \simeq 1/2$ bit.

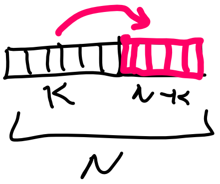
$R_c = \frac{1}{2}$



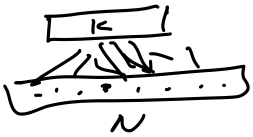


$\boxed{R_c = \frac{1}{2}}$

C



$\Rightarrow$

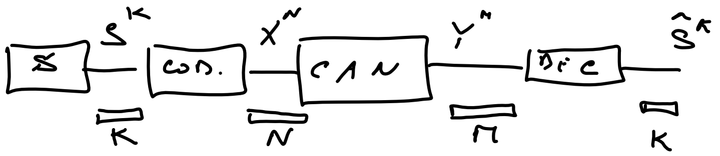


$N = 100$

2^{1000}

$N \to \infty$

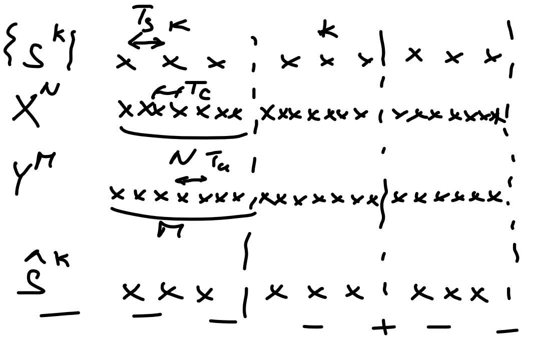
$\boxed{p(e) \simeq 0}$!!!



$S \in A = \{a_1 \dots a_d\}$

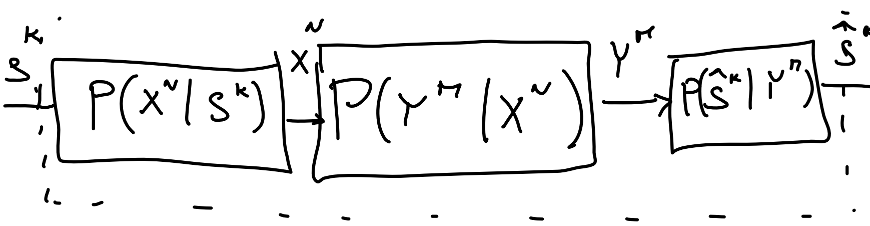
$X \in \{x_1 \dots x_m\}$

$Y \in \{y_1 \dots y_m\}$



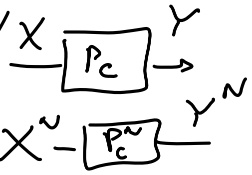
$KT_s = NT_c = MT_u$

$$R = \frac{\log_2 d^K}{KT_s} = \frac{\cancel{K}\log_2 d}{\cancel{K}T_s}$$

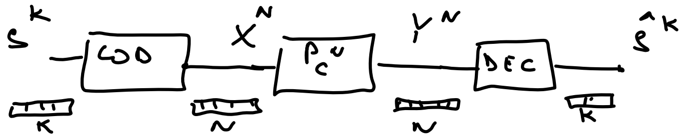


$$\frac{\log_2 m^N}{NT_c} = \frac{\log_2 m}{T_c}$$

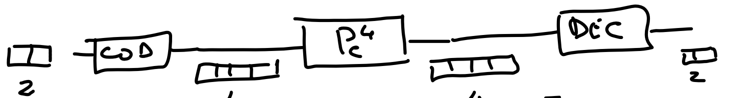
$M = N$

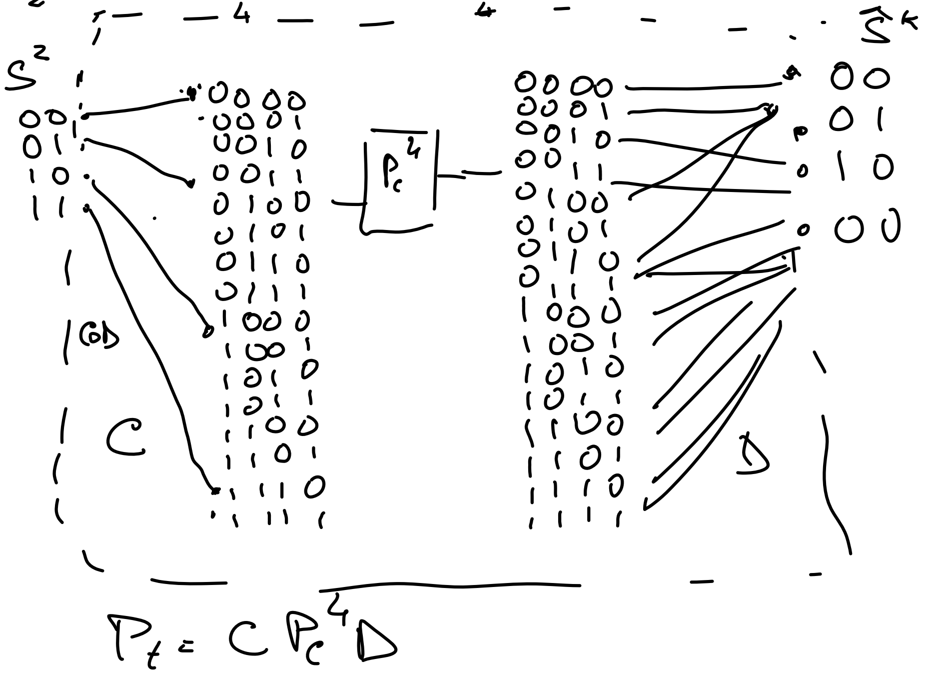


$M = N$ binario

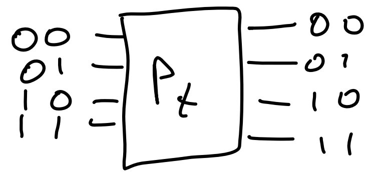


Esempio $K = 2$, $d = 2$, $N = 4$

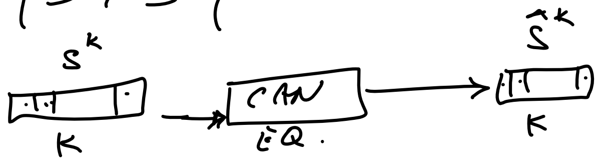




$P_t = C P_c^4 D$



$$P_{ew} = P_r\{\hat{S}^k \neq S^k\} = 1 - P_r\{\hat{S}^k = S^k\}$$

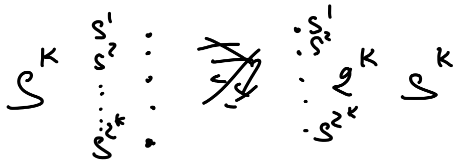


00101

00100

00111

P_r {errore di parola} $\longleftrightarrow$ P_r {errore per bit}



$$S^k \in \{s^1, s^2, \ldots\ldots, s^{2^k}\}$$
$$\hat{S}^k \in \{\,\text{''} \quad \text{''} \qquad \text{''}\,\}$$

$$P_r\{\hat{S}^k = s^i \mid S^k = s^j\}$$

$$P_{ew} = 1 - P_{cw} = 1 - P_r\{\hat{S}^k = S^k\}$$
$$= 1 - \sum_{i=1}^{2^k} P_r\{\hat{S}^k = s^i \mid S^k = s_i\} P_r\{S^k = s_i\}$$

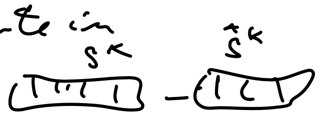
se le prob. a priori sono uguali 2^k

$$P(S^k = s_i\} = 2^{-k}$$

$$= 1 - \sum_{i=1}^{2^k} P_r\{\hat{S}^k = s^i \mid S^k = s^i\}\, 2^{-k}$$

P_b = Prob. di errore a livello di bit.

$\Rightarrow$ Quanti bit sono mediamente in errore $\Rightarrow$ $\dfrac{\#\text{ bit diversi}}{K}$



$$P_b = \frac{E[\#\text{ bit in errore}]}{K} = \frac{E[d_H(\hat{S}^k, S^k)]}{K}$$

$$= \frac{1}{K}\sum_{i=1}^{2^k}\sum_{j=1}^{2^k} d_H(\hat{S}^k = s^i, S^k = s^j)\, P_r\{\hat{S}^k = s^i ; S^k = s^j\}$$

$$= \frac{1}{K}\sum^{2^k}\sum^{2^k} d_H(\hat{S}^k = s^i, S^k = s^j)\, P_r\{\hat{S}^k = s^i \mid S^k = s^j\}\,\underbrace{P_r\{S^k = s^j\}}$$

$$= \frac{1}{K} \sum_{i=1}^{2^K} \sum_{j=1}^{2^K} d_H(\hat{S}^K = s^i, S^K = s^j) \, P_r\{\hat{S}^K = s^i | S^K = s^j\} \underbrace{P_r\{S^K = s^j\}}_{2^{-K}}$$

Esempio $K = 3$ d_H

0 0 0	0 0 0
0 0 1	0 0 1
0 1 0	0 1 0
0 1 1	0 1 1
1 0 0	1 0 0
1 0 1	1 0 1
1 1 0	1 1 0
1 1 1	1 1 1

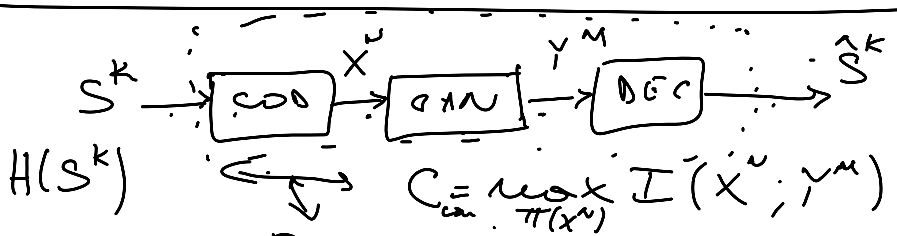
d_H	000	001	010	011	100	101	110	111
000	0	1	1	2	1	2	2	3
001	1	0	2	1				
010			0					
011				0				
100					0			
101						0		
110							0	
111								0

D_H

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_b = \frac{2^{-K}}{K} \underline{e}^T (D_H \odot P_t) \underline{e}$$

$$X^N \longrightarrow \overset{\downarrow}{\boxed{CAN}} \longrightarrow Y^N$$



$H(S^K)$ $\longleftrightarrow$ $C_{can} = \max_{\pi(X^N)} I(X^N; Y^M)$

$$P_{ew} = P_r\{S^K \neq \hat{S}^K\}$$

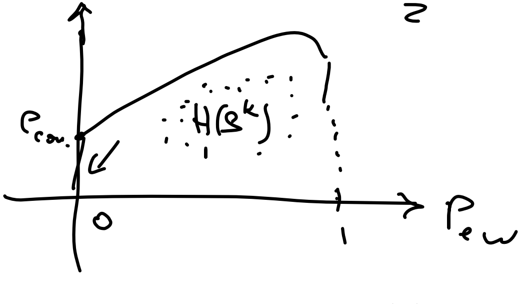
TEOREMA INVERSO DELLA CODIFICA

$$\underbrace{H(S^K)}_{} \leq \underbrace{C_{can}}_{} + H(P_{ew}) + P_{ew} \log_2 (d^K - 1)$$

$$H(P_{ew}) = P_{ew} \log_2 \frac{1}{P_{ew}} + (1 - P_{ew}) \log_2 \frac{1}{1 - P_{ew}}$$

$P_{ew} = 0$

$H(S^K) \leq C_{can}$



$$H(S^K | \hat{S}^K) \leq H(P_e) + P_e \log (d^K - 1)$$

$$H(S^k | \hat{S}^k) \leq H(P_{ew}) + P_{ew} \log_2 (d^k - 1)$$

$$I(S^k ; \hat{S}^k) = H(S^k) - H(S^k | \hat{S}^k)$$

$$I(S^k ; \hat{S}^k) \leq I(X^N ; Y^M) \leq C_{can}$$

$$H(S^k) - H(S^k | \hat{S}^k) \leq C_{can}$$

$$H(S^k) \leq C_{can} + H(S^k | \hat{S}^k)$$

$$\boxed{H(S^k) \leq C_{can} + H(P_{ew}) + P_{ew} \log_2 (d^k - 1)}$$

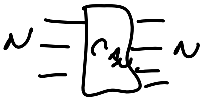
Caso particolare $P_{ew} = 0$

$$H(S^k) \leq C_{can}$$

simboli di sorgente indipendenti

$$K H(S) \leq C_{can}.$$

$N = M$



$P_C^N = P_C \otimes P_C \otimes \dots \otimes P_C$

C capacità del singolo uso del canale.

$$K H(S) \leq N C$$

$$H(S) \leq \frac{N}{K} C$$

sorgente binaria a simboli i.i.d. e uniformi $H(S) = 1$

canale binario a usi indipendenti:

$$1 \leq \frac{N}{K} C$$

$$\boxed{\frac{K}{N}} \leq C$$

Rapporto di codifica

canale: BSC P_e

K N

canale: BSC p_e

$$C = 1 - H(p_e)$$

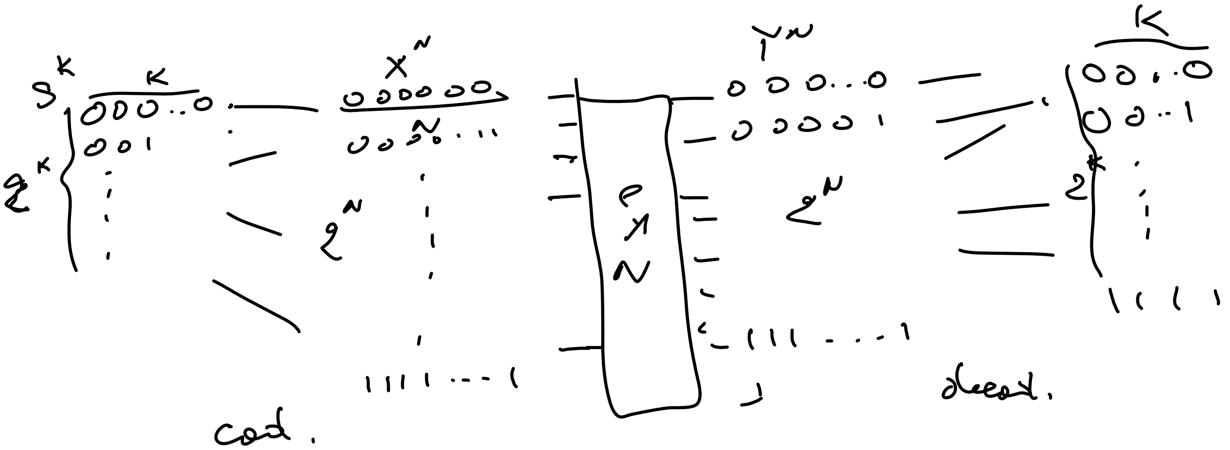
$$\frac{K}{N} \le \underbrace{1 - H(p_e)}_{0.53}$$

Es. $p_e = 0.1$

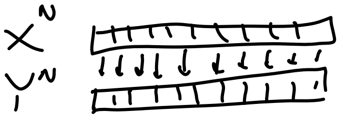
$$\frac{K}{N} \le 0.53$$

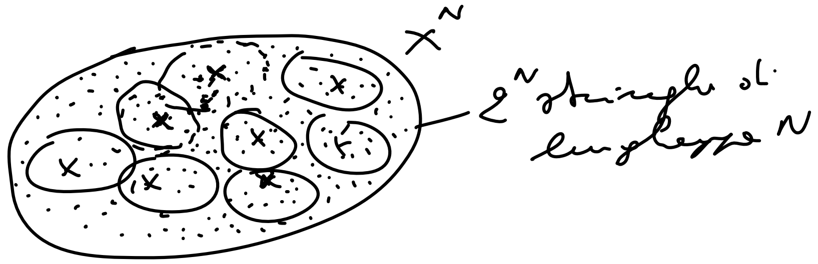
00 0000

000 000000



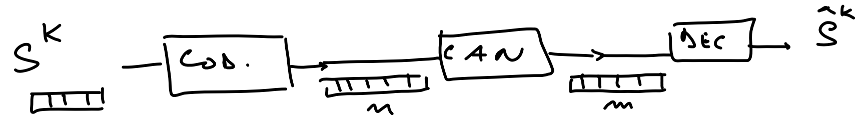
CAN: $(BSC)^N$





Lezione 12

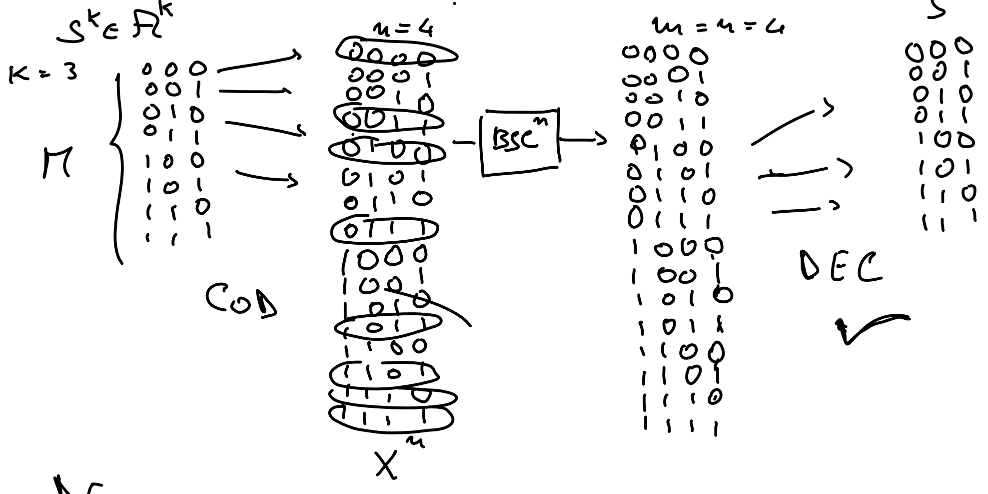
venerdì 13 novembre 2020 11.00



$R_c = \frac{k}{n}$ RAPPORTO DI CODIFICA

caso binario

$\Omega = \{0,1\}$



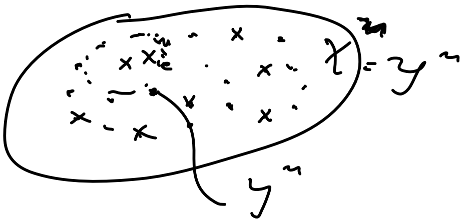
DEC : MAP ≡ Min. $P(e)$

$\pi(S^k)$ Uniforme.

CANALE BSC^n

$\mathcal{S}^k = \{s_1^k, s_2^k, \ldots, s_{2^k}^k\}$ $\quad \mathcal{X}^n = \{x_1^n, x_2^n, \ldots \ldots x_{2^n}^n\}$

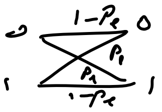
$\mathcal{X}_c^n = \{x_{1c}^n, x_{2c}^n, \ldots x_{2^kc}^n\}$



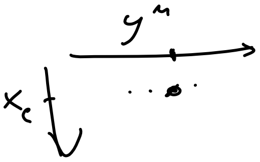
$\mathcal{Y}^n = \mathcal{X}^n$

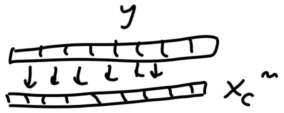
MAP → MV

Teo: Regola ottima è quella a minima distanza di Hamming



$$P_2(Y^n = y^n \mid X^n = x_c^n)$$





$P_e < 1/2$

$$= P_e^{d_H(y^n, x_c^n)} (1-p_e)^{(n - d_H(y^n, x_c^n))}$$

$$Q = p^d (1-p)^{n-d} \qquad p < \frac{1}{2}$$

$$d_1 < d_2$$

$$p^{d_1} (1-p)^{n-d_1} > p^{d_2} (1-p)^{n-d_2}$$

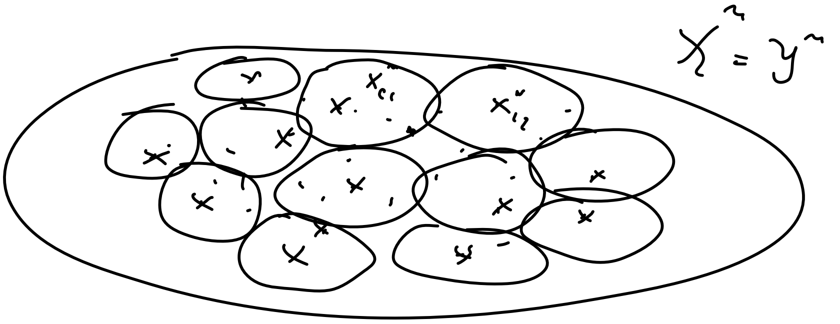
$$p^{d_1-d_2} \quad (\quad)^{\cancel{n}-d_2-\cancel{n}+d_1}$$

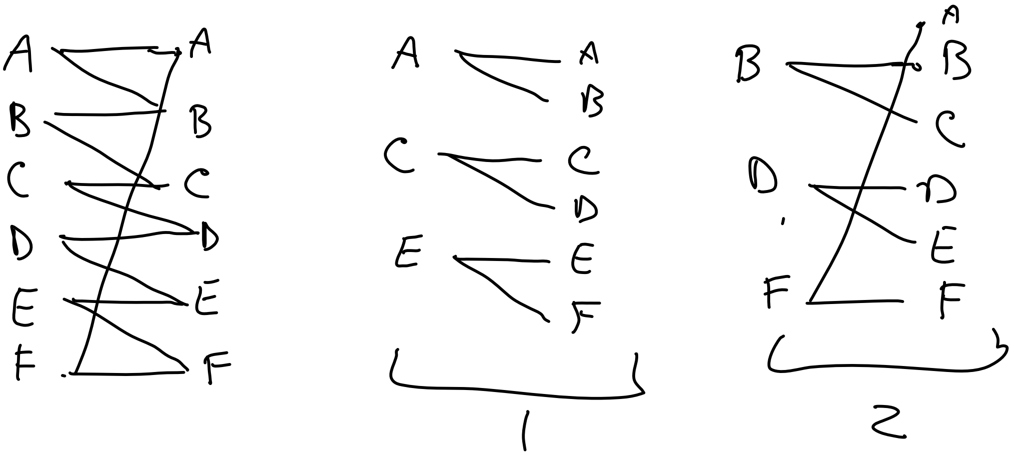
$$p^{d_1-d_2} > (1-p)^{\cancel{n}-d_2-\cancel{n}+d_1}$$

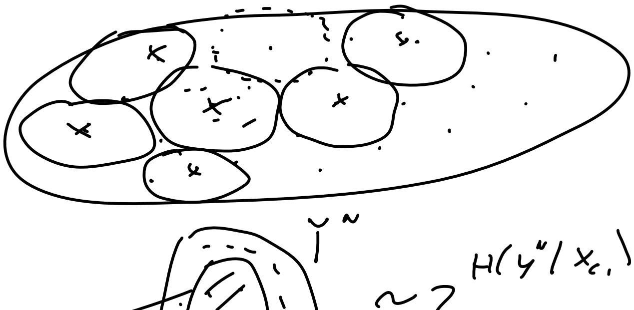
$$p^{d_1-d_2} > (1-p)^{d_1-d_2}$$

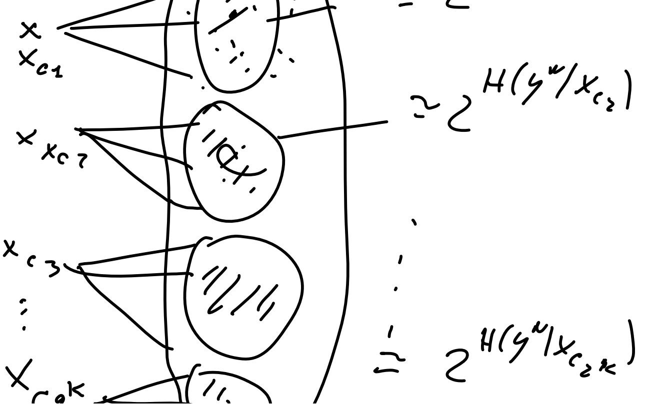
$$\left(\frac{p}{1-p}\right)^{d_1-d_2} > 1$$

. . . . .









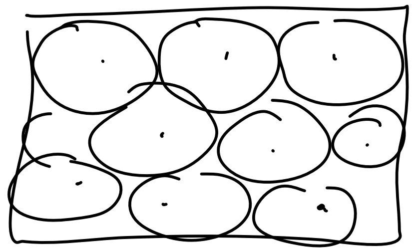
→ Dimensione media di ogni sfera

$2^{H(Y^n/X^n)}$

$$\doteq 2^{H(y^N/x_{c2^K})}$$

Dimensione di Y^N

$$\simeq 2^{H(Y^N)}$$



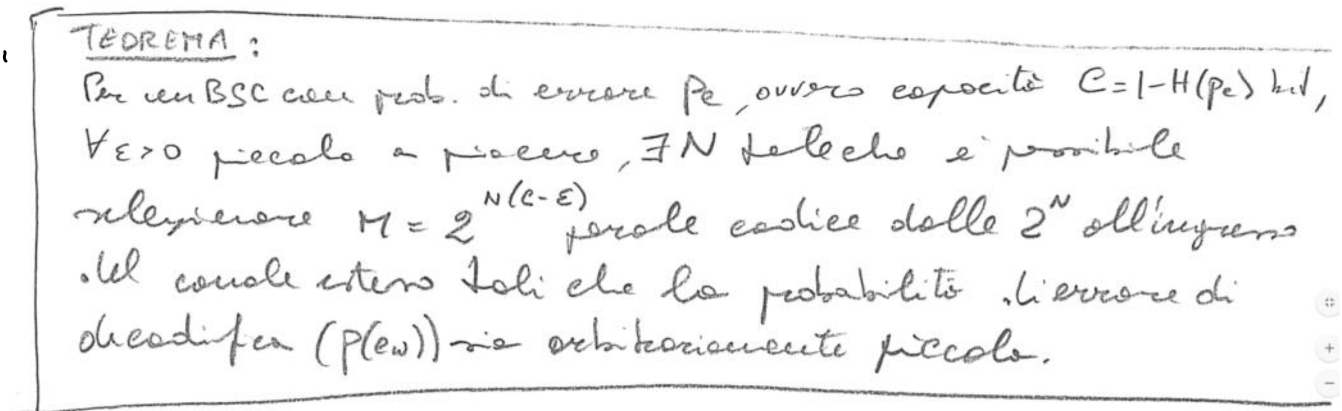
$$\frac{2^{H(Y^N)}}{2^{H(Y^N/X^N)}} \doteq \text{\# di sfere non sovrapposte}$$

$= \#$ di parole codice.

$$2^{\underbrace{H(Y^N)-H(Y^N/X^N)}} = 2^{I(X^N;Y^N)} < 2^{CN}$$

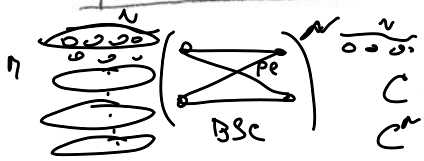
$$C \quad R^K \simeq 2^C$$

TEOREMA DELLA CODIFICA (o II T. DI SHANNON)



TEOREMA:

Per un BSC con prob. di errore p_e, ovvero capacità $C = 1 - H(p_e)$ bit, $\forall \varepsilon > 0$ piccolo a piacere, $\exists N$ tale che è possibile selezionare $M = 2^{N(C-\varepsilon)}$ parole codice dalle 2^N all'ingresso del canale esteso tali che la probabilità di errore di decodifica $(P(e_w))$ sia arbitrariamente piccola.



$C = 1 - H(p_e)$

$C^N = N(1 - H(p_e))$

DIMOSTRAZIONE

Dec ottimo $\Longleftrightarrow$ Min distanza di Hamming

$\Downarrow$

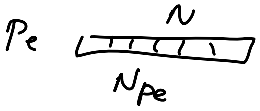
Dec. sub-ottimo $\Rightarrow$ funziona lo stesso

Prova dipendente dal codice specifico $\Longrightarrow$ Prova su un codice aleatorio

$\Downarrow$

Mediamente tutti i codici funzionano (T. della cod canali)

Dec - SUB-OTTIMA (Decodifica a sfera, SPHERE-DECODING)



$$P(\cup A_i) \le \sum_{i=1}^{N} P(A_i)$$

$$p(e_w | X^N = x^N_{c_j}) = P_j(e_w)$$

$$P_j(e_w) = \underbrace{P_r\{x^N_{c_j} \notin Y^\varepsilon\}}_{I} + \underbrace{P_r\{x^N_{c_j} \in Y^\varepsilon\}}_{\le 1} \cdot P_r\left\{\begin{matrix}\text{almeno un'altra} \\ \text{parola codice} \in Y^\varepsilon\end{matrix}\right\}$$

$$P_j(e_w) \le P_r\{x^N_{c_j} \notin Y^\varepsilon\} + P_r\left\{\begin{matrix}\text{almeno un'altra} \\ \in Y^\varepsilon\end{matrix}\right\}$$

bound dell'Unione.

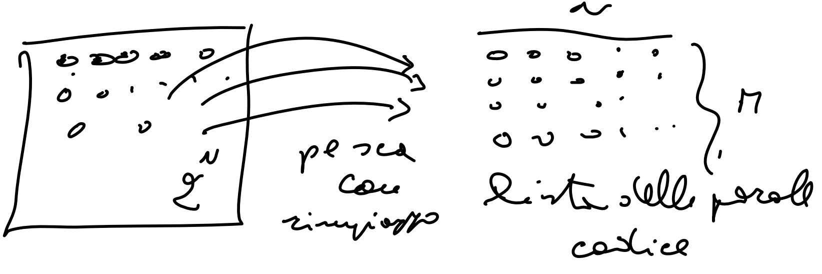
$$\le \underbrace{P_r\{x^N_{c_j} \notin Y^\varepsilon\}}_{I} + \sum_{\substack{i=1 \\ i\neq j}}^{M} P_r\{x^N_{c_i} \in Y^\varepsilon\}$$

Prob. $x^N_{c_j}$ cada all'esterno della sfera.

$N \to \infty$ → arbitrariamente piccola.

$$P_j(e_w) \le \delta + \sum_{\substack{i=1 \\ i\neq j}}^{M} P_r\{x^N_{c_i} \in Y^\varepsilon\}$$

svincolarsi dal codice specifico



CODICE ALEATORIO

$(2^N)^M$ # di possibili codici.

$3 \to \boxed{3^N}$

2^N sia sufficientemente grande tale che la prob. di scegliere la stessa stringa più volte sia trascurabile.

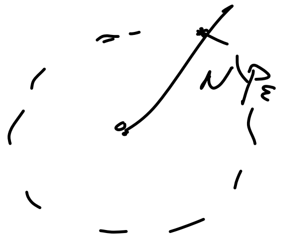
$\longrightarrow \overline{P(\ldots)} \le \delta + \sum^{M} \overline{P_r\{x^N_{c_i} \in Y^\varepsilon\}}$

lo stesso stringa più volte sia [illegible].

mediazione rispetto ai possibili codici $\xrightarrow{\quad}$ $$\overline{P_j(e_w)} \le \delta + \sum_{\substack{i=1 \\ i\ne j}}^{M} \overline{P_r\{y_{c_i}^N \in Y^\varepsilon\}}$$

$$\overline{P_j(e_w)} \le \delta + (M-1)\,\overline{P_r\{x_c^N \in Y^\varepsilon\}}$$

$$\overline{P_j(e_w)} \le \delta + M\,\overline{P_r\{x_c^N \in Y^\varepsilon\}}$$



$$P_r\{x_c^N \in Y^\varepsilon\} = \frac{N(Np_\varepsilon)}{2^N}$$

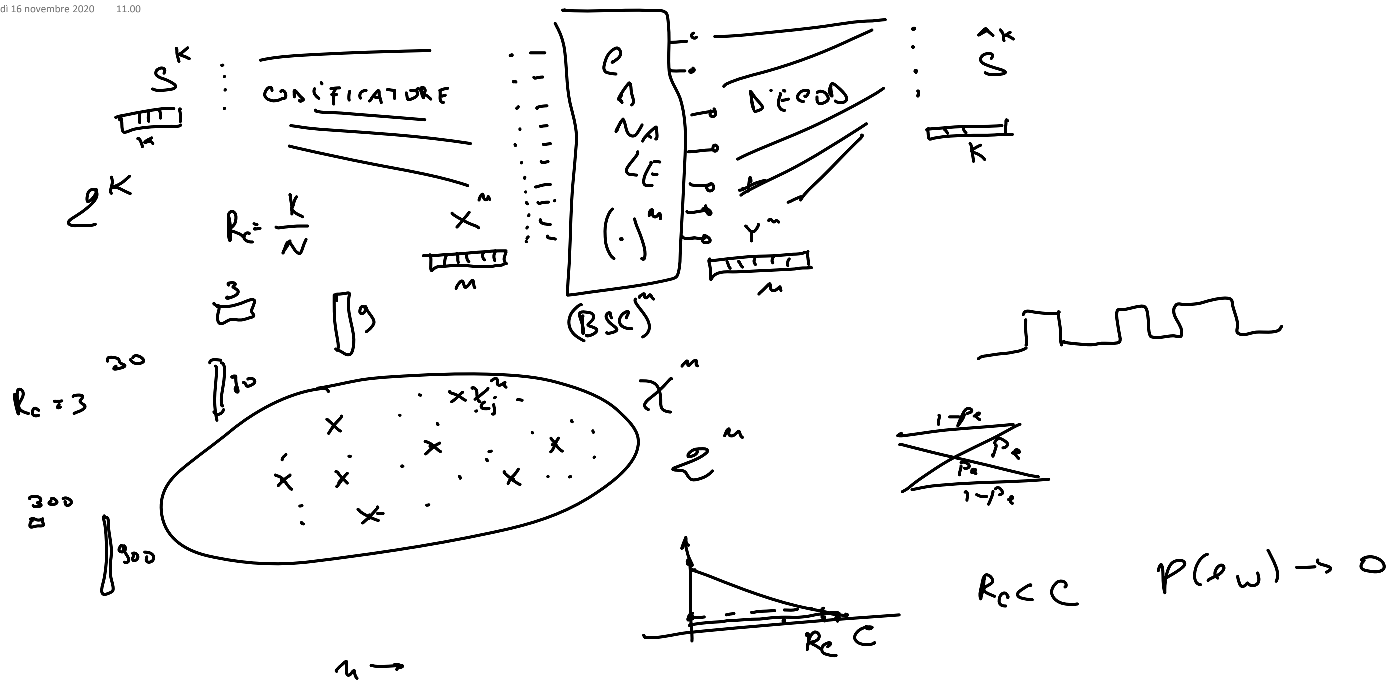
$N \to \infty$

$$M < 2^{N\underbrace{(1-H(p_e))}_{C}}$$

$$\overline{P(e_w)} \longrightarrow 0$$

$$2^K < 2^{NC}$$

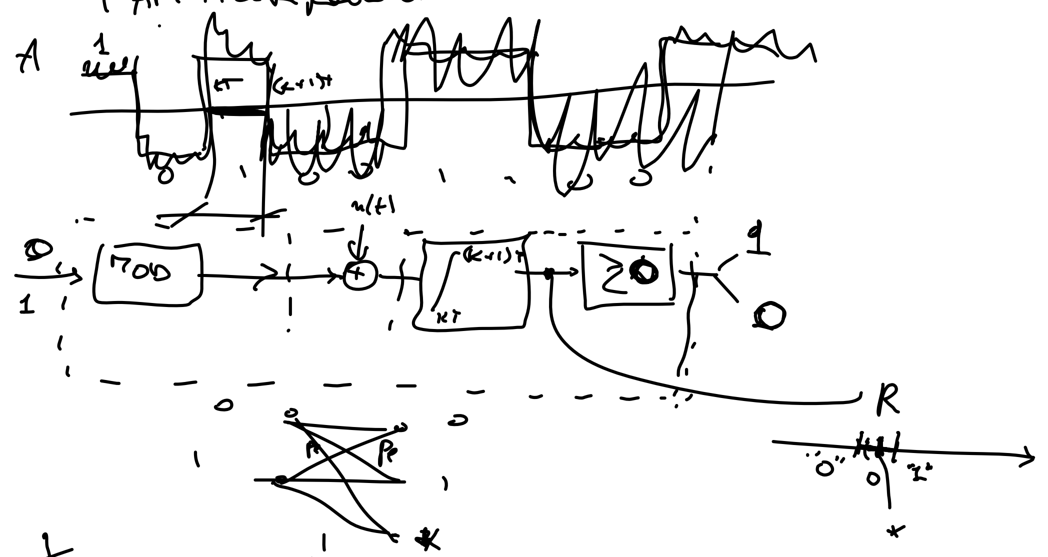
$$\boxed{\frac{K}{N} < C} \xrightarrow[N\to\infty]{} \overline{P(e_w)} \to 0$$

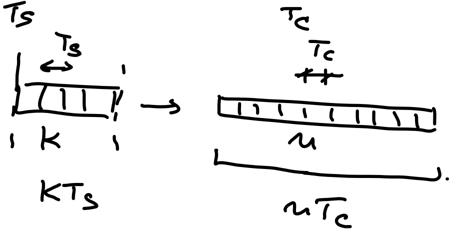


$R_c < C \quad P(e_w) \rightarrow 0$

Esempio di modulaz. numerica.

PAM Antipodale

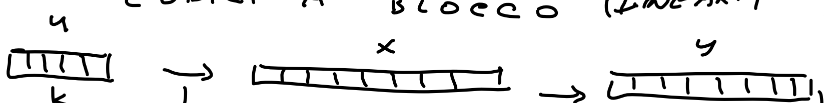




4	8
5	10
8	16

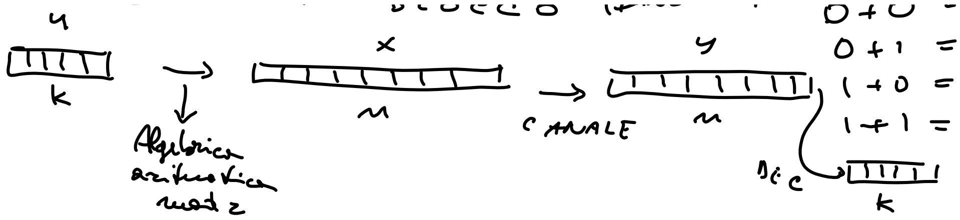
$R_c = \frac{1}{2}$

CODICI A BLOCCO (LINEARI)



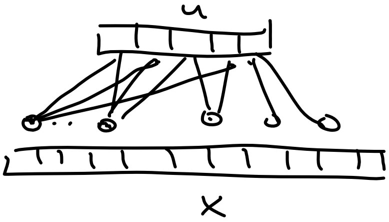
0 + 0 = 0
0 + 1 = 1
1 + 0 = 1

blocco



0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 0

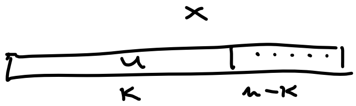
Algebrica aritmetica mod 2



Matrice generatrice

$\underline{x} = \underline{u}\, \underline{\underline{G}}$

Codice sistematico



(n, k)

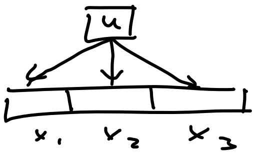
Esempio

Codice a ripetizione

$K = 1 \quad n = 3$

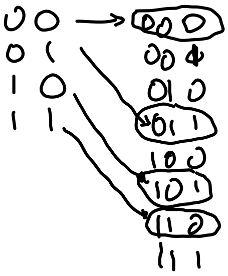
G

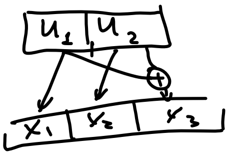
$\underline{x} = \underline{u}\, [1\ 1\ 1]$



codice a controllo di parità sistematico

$K = 2 \ , n = 3$





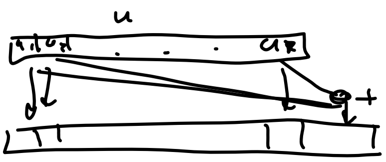
$$\begin{cases} x_1 = u_2 \\ x_2 = u_2 \\ x_3 = u_1 + u_2 \end{cases}$$

$$[x_1 x_2 x_3] = [u_1 u_2] \overbrace{\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]}^{I_k}$$

$G \quad \uparrow \underline{\underline{P}}$

Co~~dicke~~ Codice a controllo di parità

$(n, k) = (8, 7)$

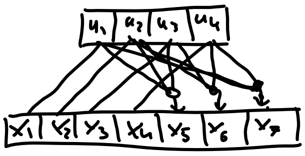


$$\begin{cases} x_i = u_i \quad i = 1, .., 7 \\ x_8 = \sum_{i=1}^{7} u_i \end{cases}$$

G

$$[x_1 \dots x_8] = [u_1 \dots u_7] \left[\begin{array}{ccc:c} 1 & & 0 & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ 0 & & 1 & 1 \end{array}\right]$$

Codice di Hamming (7,4)



$$\begin{cases} x_i = u_i \quad i = 1, .. 4 \\ x_5 = u_1 + u_2 + u_3 \\ x_6 = u_2 + u_3 + u_4 \\ x_7 = u_1 + u_2 + u_4 \end{cases}$$

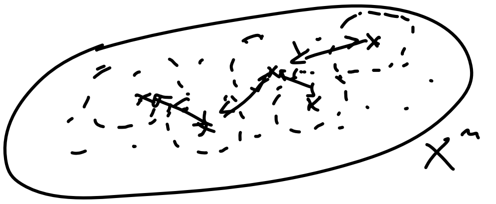
$$[x_1 x_2 x_3 x_4 x_5 x_6 x_7] = [u_1 u_2 u_3 u_4]\left[\begin{array}{cccc:ccc} 1&0&0&0&1&0&1\\ 0&1&0&0&1&1&1\\ 0&0&1&0&1&1&0\\ 0&0&0&1&0&1&1 \end{array}\right]$$

$$\boxed{\underline{\underline{G}} = [I_k \,\vdots\, P] \quad \uparrow \text{Matrice di parità}}$$

Property 1 The block code consists of all possible sums of the rows of the generator matrix.

Property 2 The sum of two code words is still a code word.

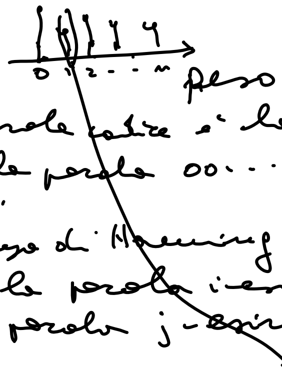
Property 3 The n-tuple of all zeros is always a code word.



Peso di una parola codice

\# bit = 1

Distribuzione dei pesi



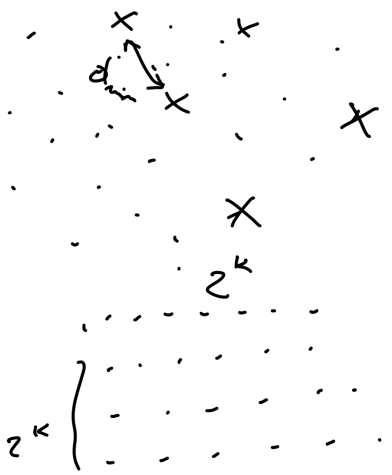
peso di una parola codice è la distanza dalla parola 00 . . . 0

$d_H(i,j)$ = distanza di Hamming tra la parola i-esima e la parola j-esima

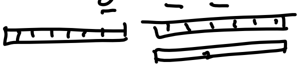
d_{min} = minimo peso

\# della parola nulla

d_{min}



$\underline{y} = \underline{x} + \underline{e}$



0 1

1 1

$$\underset{n}{\underline{x}} = \underset{k}{\underline{u}}\;\underline{\underline{G}} \qquad k\,\begin{array}{c}\begin{array}{cc}k & n-k\end{array}\\ \boxed{\underline{\underline{I}}_k \,\vdots\, \underline{P}}\end{array}\,k$$

SINDROME $\underline{S} = \underline{y}\,\underline{\underline{H}}' = \overline{n-k}$

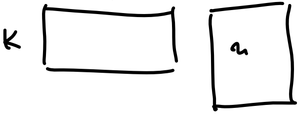
$$\underset{n-k}{\boxed{\cdots}} = \underset{y}{\boxed{}} \begin{bmatrix} P \\ \hdashline I_{n-k} \end{bmatrix} \begin{array}{l} k \\ n-k \end{array}$$

$$\underline{\underline{H}} = [\underset{k}{\underline{\underline{P}}^T} \,\vdots\, \underset{n-k}{I_{n-k}}]$$

$$n-k\left[\begin{array}{c:ccc} & 1&0&0\\ P^T & 0&1&0\\ & 0&0&1 \end{array}\right]$$

MATRICE DI CONTROLLO DI PARITA'

$$\underline{\underline{G}}\,\underline{\underline{H}}' = \underline{0}$$



$$k\underset{G}{\overset{n}{[I_k \,\vdots\, P]}}_n \overset{n-k}{\begin{bmatrix} P \\ I_{n-k} \end{bmatrix}} = k\overset{n-k}{\boxed{\underline{0}}}$$

0 1 1 0

Esempio

H(7,4)

$$\begin{cases} x_1 = u_1 \\ x_2 = u_2 \\ x_3 = u_3 \\ x_4 = u_4 \end{cases} \xrightarrow{\text{canale}} \begin{array}{l} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \quad \begin{cases} S_1 = (y_1 + y_2 + y_3) + y_5 \\ S_2 = (y_2 + y_3 + y_4) + y_6 \\ S_3 = (y_1 + y_2 + y_4) + y_7 \end{cases}$$

$$\begin{cases} x_3 = u_3 \\ x_4 = u_4 \\ x_5 = u_1 + u_2 + u_3 \\ x_6 = u_2 + u_3 + u_4 \\ x_7 = u_1 + u_2 + u_4 \end{cases} \xrightarrow{\text{arrivo}} \begin{matrix} y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{matrix}$$

$$(s_3 = (y_1 + y_2 + y_4) + y_7$$

$$[s_1\ s_2\ s_3] = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{s} = \underline{y} \underbrace{\begin{bmatrix} \underline{\underline{P}} \\ \\ \underline{I}_{n-k} \end{bmatrix}}_{H'}$$

$\underline{S} = \underline{0}$ no problem?

$\underline{S} \neq \underline{0}$ qualche problema certamente $\Big] \rightarrow$ rigetto / correggerla

2: Standard array of the (7, 4) Hamming code.

Syndromes	Error patterns															
	Coset leaders	Other errors														
000	0000000	1000101	0100111	0010110	0001011	1100010	1010011	1001110	0110001	0101100	0011101	0111010	1011000	1101001	1110100	1111111
001	0000001	1000100	0100110	0010111	0001010	1100011	1010010	1001111	0110000	0101101	0011100	0111011	1011001	1101000	1110101	1111110
010	0000010	1000111	0100101	0010100	0001001	1100000	1010001	1001100	0110011	0101110	0011111	0111000	1011010	1101011	1110110	1111101
011	0001000	1001101	0101111	0011110	0000011	1101010	1011011	1000110	0111001	0100100	0010101	0110010	1010000	1100001	1111100	1110111
100	0000100	1000001	0100011	0010010	0001111	1100110	1010111	1001010	0110101	0101000	0011001	0111110	1011100	1101101	1110000	1111011
101	1000000	0000101	1100111	1010110	1001011	0100010	0010011	0001110	1110001	1101100	1011101	1111010	0011000	0101001	0110100	0111111
110	0010000	1010101	0110111	0000110	0011011	1110010	1000011	1011110	0100001	0111100	0001101	0101010	1001000	1111001	1100100	1101111
111	0100000	1100101	0000111	0110110	0101011	1000010	1110011	1101110	0010001	0001100	0111101	0011010	1111000	1001001	1010100	1011111

2^{n-k} $\qquad 2^{n-k}$

2

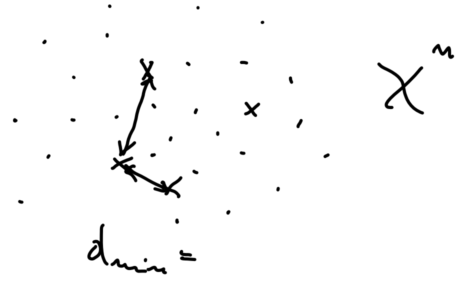
2^7 parole codice

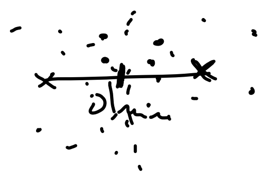
$\underline{y} = \underline{x} + \boxed{\underline{e}}$

$p = 0.01 \quad \left(\frac{1}{100}\right)^3 = \frac{1}{10^6}$

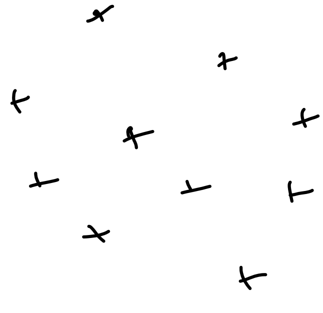
p^3

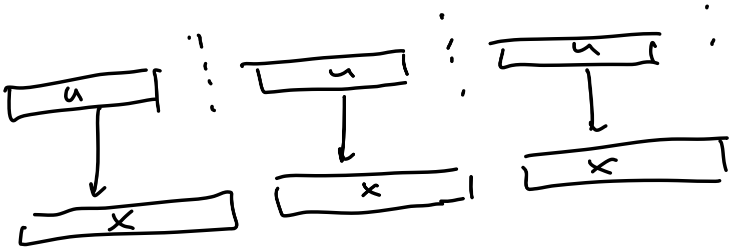
$d_H(\underline{y}, \underline{x}_e)$ $\qquad \underline{y} + \underline{e} = \underline{x} \longrightarrow \underline{u}$

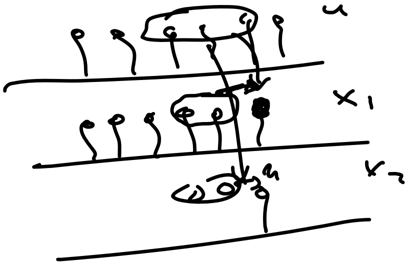


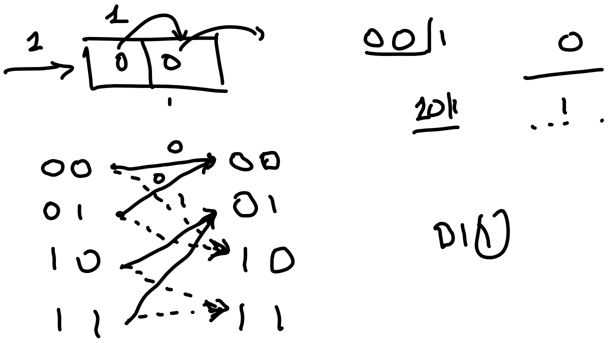


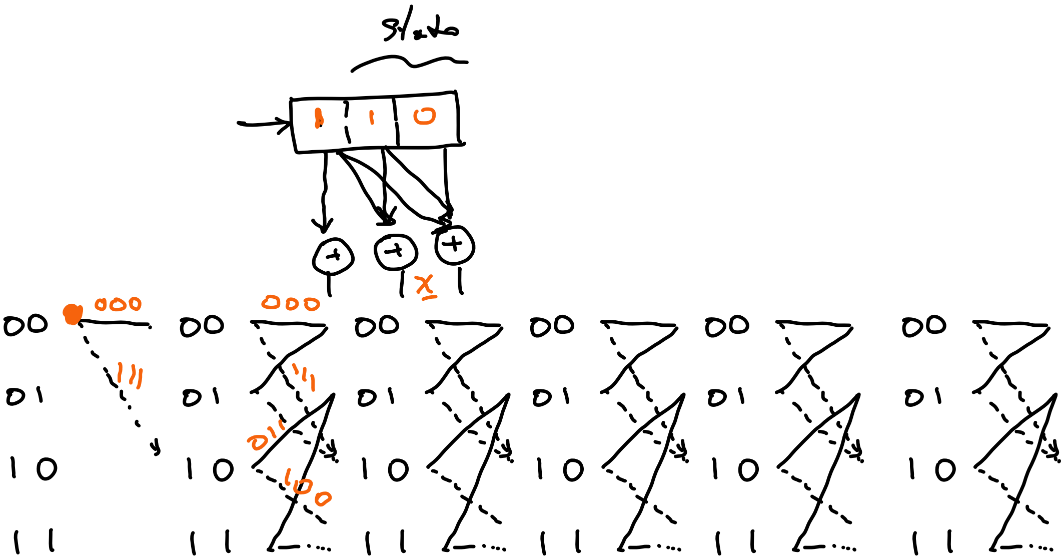
Codici a blocco con dmin grande.



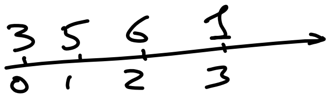








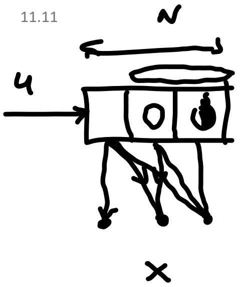
z-trasformata



$$3+5z^{-1}+6z^{-2}+1z^{-3}$$

Lezione 15

lunedì 23 novembre 2020 11.11

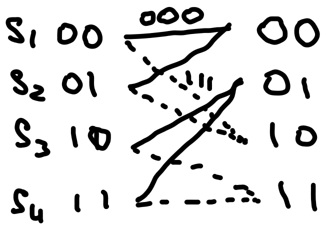


1 k N

(3,1,3)

$\frac{K}{M}$

101

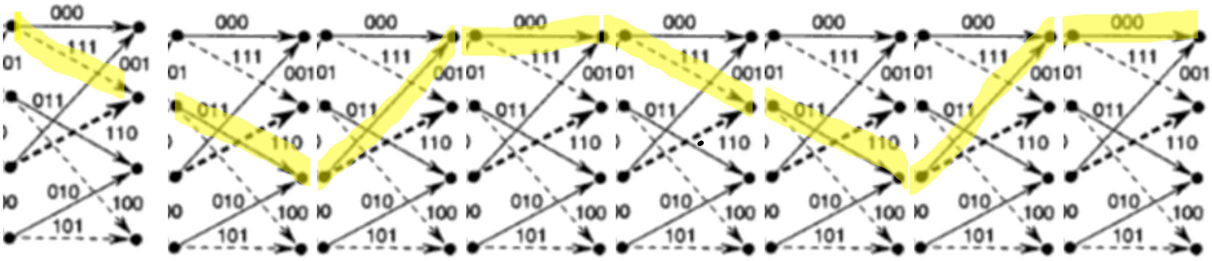


1 1 0 1 1 . . . .

111 100 010 110 100

⇓ canale (errori multipli)

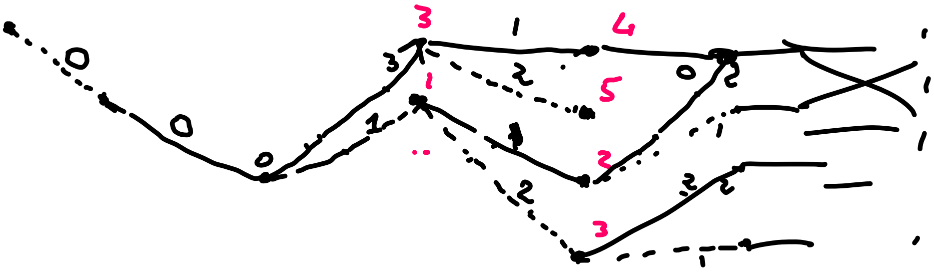
1 0 0 0 1 0 0 0



111 011 001 000 111 011 001 000 x

⇓ errori.

111 011 100 010 111 111 001 000

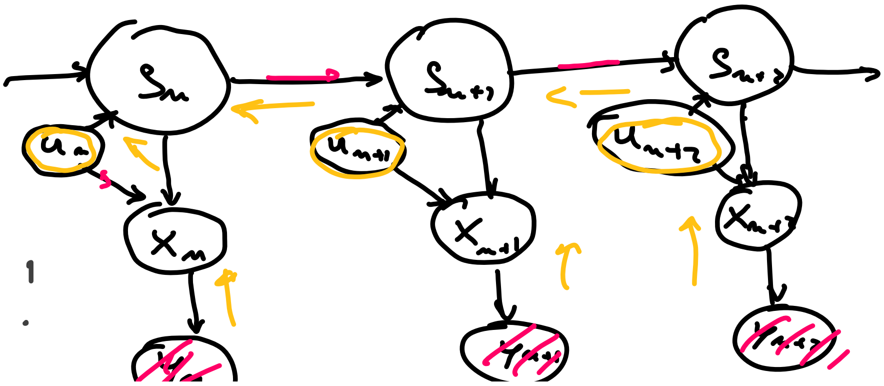


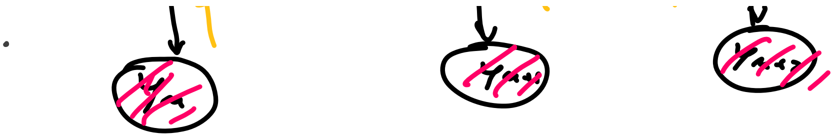
Hard

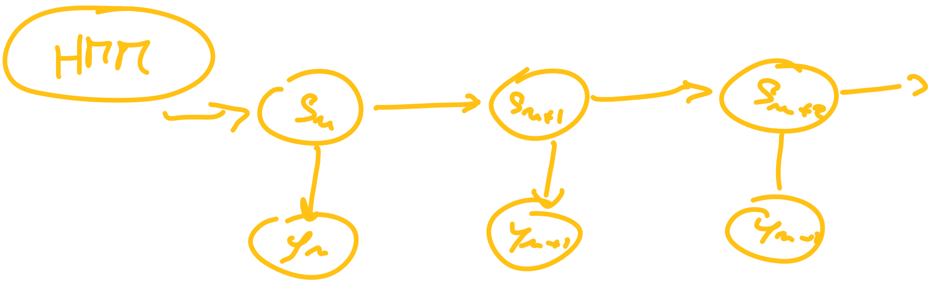
Soft

MODELLO BAYESIANO

Propagazione delle probabilità




y_n
y_{n+1}
y_{n+2}


HMM
S_n
S_{n+1}
S_{n+2}
y_n
y_{n+1}
y_{n+2}

Lezione 16

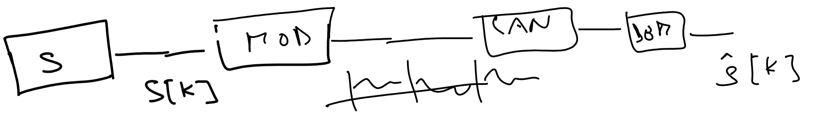
venerdì 27 novembre 2020 10:57

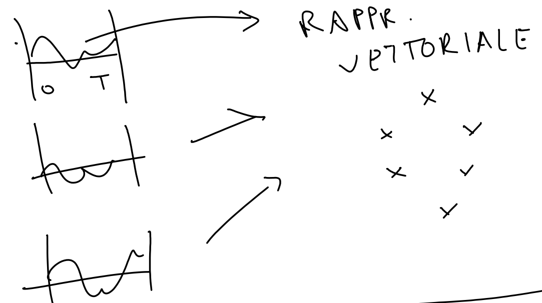
Introduzione alla modulaione numerica

Introduzione allo spazio dei segnali

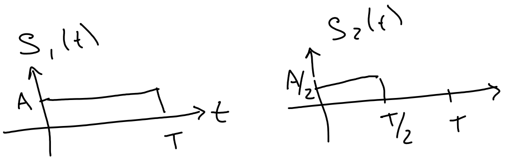
Lezione 17

lunedì 30 novembre 2020 10.59





ESEMPIO



$$\int_0^T s_1(t)s_2(t)\,dt \neq 0$$

non ortogonali.

$$c_1 s_1(t) + c_2 s_2(t) = 0$$

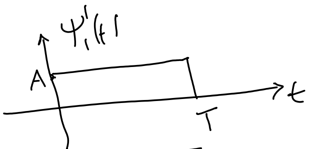
linearmente ind.

$$\|\cdot\|^2 = \int_0^T (\cdot)^2\,dt$$

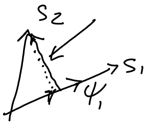
Cerco base ortonormale di rappresentazione

Proc. di Gram-Schmidt

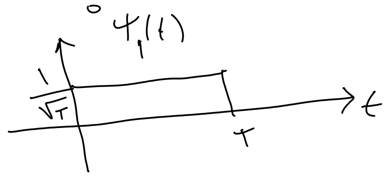
① $\psi_1'(t) = s_1(t)$



$$\psi_1(t) = \frac{\psi_1'(t)}{\sqrt{\|\psi_1'\|^2}} \qquad \|\psi_1'\|^2 = \int_0^T \psi_1'(t)^2\,dt = A^2T$$



$$\psi_1(t) = \frac{\psi_1'(t)}{A\sqrt{T}}$$



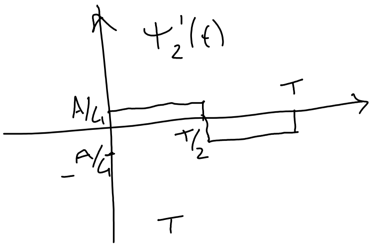
✓

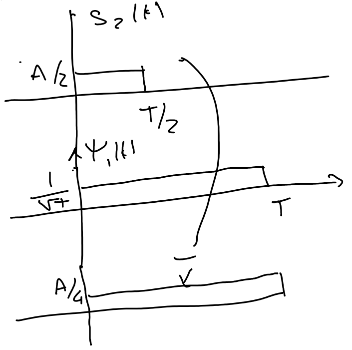
② $\psi_2'(t) = s_2(t) - \langle s_2, \psi_1 \rangle \psi_1(t)$

$$\langle s_2, \psi_1 \rangle = \int_0^T s_2(t)\psi_1(t)\,dt$$

$$= \int_0^{T/2} \frac{A}{2}\frac{1}{\sqrt{T}}\,dt = \frac{A}{2\sqrt{T}}\frac{T}{2} = \frac{A\sqrt{T}}{4}$$

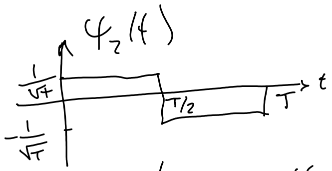
$$= s_2(t) - \frac{A\sqrt{T}}{4}\psi_1(t)$$





$$\|\psi_2'\|^2 = \int_0^T \frac{A^2}{16}\,dt = \frac{A^2T}{16}$$

$$\psi_2(t) = \frac{\psi_2'(t)}{\sqrt{\|\psi_2'\|^2}} = \frac{\psi_2'(t)\,4}{A\sqrt{T}}$$



Rappresento $s_1(t)$ e $s_2(t)$ nella base $(\psi_1(t), \psi_2(t))$

$$s_1(t) \longrightarrow \langle s_1, \psi_1 \rangle \psi_1(t) + \langle s_1, \psi_2 \rangle \psi_2(t)$$

$$s_2(t) \longrightarrow \langle s_2, \psi_1 \rangle \psi_1(t) + \langle s_2, \psi_2 \rangle \psi_2(t)$$

$$\int^T \qquad \int^T \qquad = A\,T = A\sqrt{T}$$

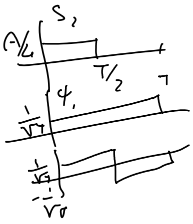
$$s_2(t) \longrightarrow \langle s_2,\psi_1\rangle \psi_1(t) + \langle s_2,\psi_2\rangle \psi_2(t)$$

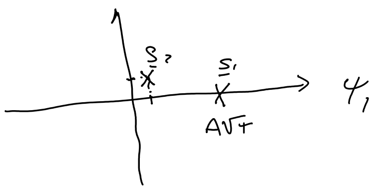
$$\langle s_1,\psi_1\rangle = \int_0^T s_1(t)\psi_1(t)\,dt = \int_0^T A\frac{1}{\sqrt{T}}dt = \frac{A}{\sqrt{T}}T = A\sqrt{T}$$

$$\langle s_1,\psi_2\rangle = \int_0^T s_1(t)\psi_2(t)\,dt = 0$$

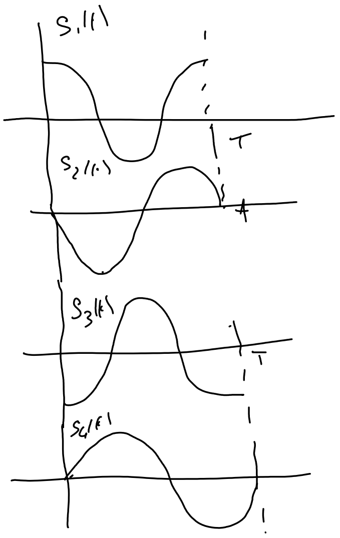
$$\langle s_2,\psi_1\rangle = \int_0^{T/2}\frac{A}{4}\frac{1}{\sqrt{T}}\,dt = \frac{A}{4}\frac{1}{\sqrt{T}}\frac{T}{2} = \frac{A\sqrt{T}}{8}$$

$$\langle s_2,\psi_2\rangle = \int_0^{T/2}\frac{A}{4}\frac{1}{\sqrt{T}} = \frac{A}{4\sqrt{T}}\frac{T}{2} = \frac{A\sqrt{T}}{8}$$





$t \in [0,T]$



$$\begin{cases} \psi_1(t) = \sqrt{\frac{2}{T}}\cos\frac{2\pi}{T}t \\ \psi_2(t) = -\sqrt{\frac{2}{T}}\sin\frac{2\pi}{T}t \end{cases}$$

$$\psi_3'(t) = s_3(t) - \underbrace{\langle s_3,\psi_1\rangle}\psi_1(t) - \langle s_3,\psi_2\rangle \psi_2(t)$$

$$= 0$$

$$\psi_4'(t) = s_4(t) - \langle s_4,\psi_1\rangle\psi_1(t) - \langle s_4,\psi_2\rangle\psi_2(t)$$

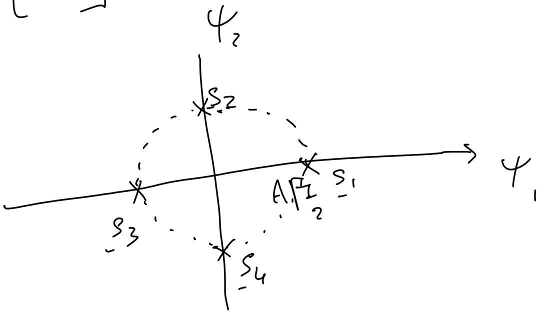
$$\langle s_4,\psi_3\rangle \psi_3(t)$$

$$\underline{s}_1 = \begin{bmatrix}\langle s_1,\psi_1\rangle \\ \langle s_1,\psi_2\rangle\end{bmatrix} = \begin{bmatrix}\int_0^T s_1(t)\psi_1(t)\,dt = \int_0^T A\cos\frac{2\pi}{T}t\sqrt{\frac{2}{T}}\cos\frac{2\pi}{T}t\,dt = A\sqrt{\frac{2}{T}}\int_0^T \cos^2\frac{2\pi}{T}t\,dt \\ 0\end{bmatrix}$$

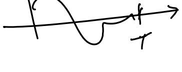
$$\frac{1}{2}+\frac{1}{2}\cos\frac{4\pi}{T}t$$

$$= \begin{bmatrix}A\sqrt{\frac{T}{2}} \\ 0\end{bmatrix} \qquad = A\sqrt{\frac{2}{T}\frac{T}{2}} = A\sqrt{\frac{T}{2}}$$

$$\underline{s}_2 = \begin{bmatrix}\langle s_2,\psi_1\rangle \\ \langle s_2,\psi_2\rangle\end{bmatrix} = \begin{bmatrix}0 \\ A\sqrt{\frac{T}{2}}\end{bmatrix} \qquad \underline{s}_3 = \begin{bmatrix}\langle s_3,\psi_1\rangle \\ \langle s_2,\psi_2\rangle\end{bmatrix} = \begin{bmatrix}-A\sqrt{\frac{T}{2}} \\ 0\end{bmatrix} \qquad \underline{s}_4 = \begin{bmatrix}0 \\ -A\sqrt{\frac{T}{2}}\end{bmatrix}$$

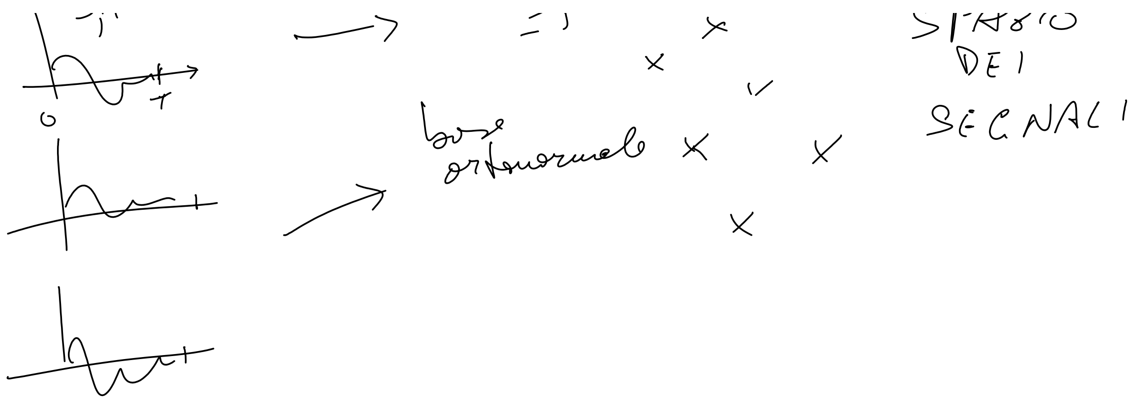


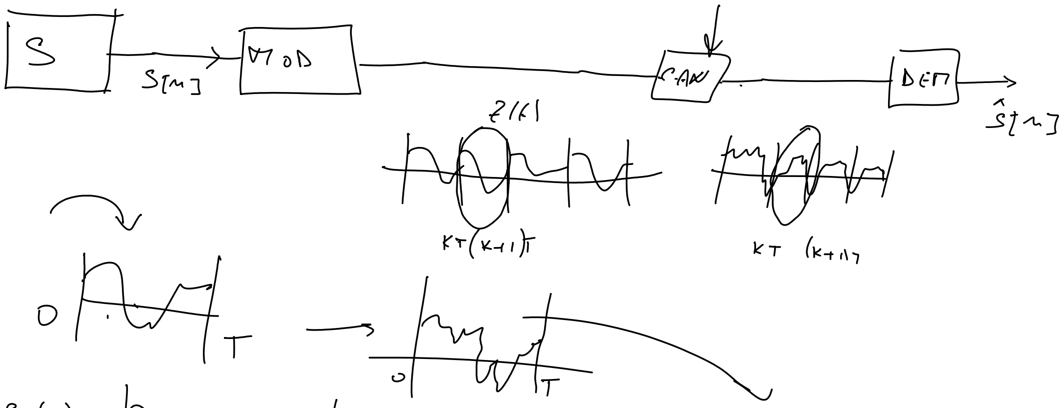
PSK



$\longrightarrow \quad \underline{s}_j$

SPAZIO
DEI



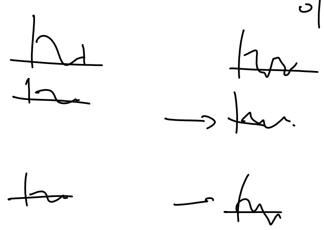


$a_1 \rightarrow s_1(t)$

$a_2 \rightarrow s_2(t)$

M simboli

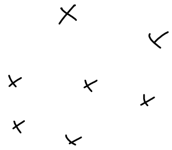
$a_M \rightarrow s_M(t)$



$$z(t) = s_j(t) + n(t)$$

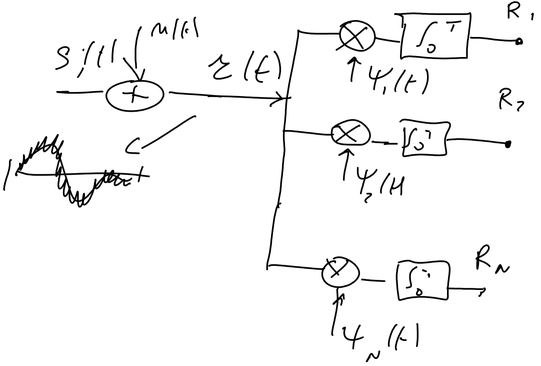
$\Downarrow$

DESCRITTI
IN UNA BASE ORTONORMALE
COMPLETA PER TUTTI I SEGNALI



$$\{\psi_1(t), \psi_2(t), \ldots \psi_N(t)\} \quad N \leq M$$

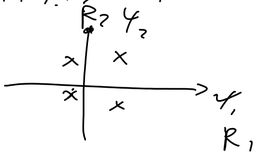
Non ~~c'è rumore~~



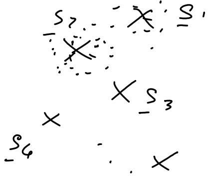
ESEMPIO

M = 4 , N = 2

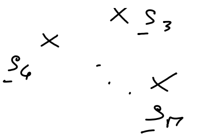
$\{\psi_1(t), \psi_2(t)\}$



$$\underline{R}(a_j) = \begin{bmatrix} \int_0^T s_j(t)\psi_1(t)dt \\ \int_0^T s_j(t)\psi_2(t)dt \\ \vdots \end{bmatrix} = \underline{s}_j = \begin{bmatrix} s_{j1} \\ s_{j2} \\ \vdots \\ s_{jN} \end{bmatrix}$$



$$\begin{bmatrix} \int_0 s_j(t)\varphi_2(t)dt \\ \vdots \\ \int s_j(t)\varphi_N(t)dt \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ s_{jN} \end{bmatrix}$$



CON RUMORE

$$\underline{R}/\alpha_j = \begin{bmatrix} \int (s_j(t)+n(t))\varphi_1(t)dt \\ \int (s_j(t)+n(t))\varphi_2(t)dt \\ \vdots \\ \int (s_j(t)+n(t))\varphi_N(t)dt \end{bmatrix} = \underline{S}_j + \begin{bmatrix} \int_0^T n(t)\varphi_1(t)dt \\ \int_0^T n(t)\varphi_2(t)dt \\ \vdots \\ \int_0^T n(t)\varphi_N(t)dt \end{bmatrix} = \underline{S}_j + \underline{N}$$

Ipotesi $n(t)$ gaussiano, a media nulla, bianco

$$\underline{N} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_N \end{bmatrix} \qquad E[N_i] = E\left[\int_0^T n(t)\varphi_i(t)dt\right] = 0$$

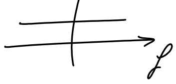
$$E[N_i^2] = E\left[\int_0^T n(t_1)\varphi_i(t_1)dt_1 \int_0^T n(t_2)\varphi_i(t_2)dt_2\right]$$

$$= \int_0^T\int_0^T E[n(t_1)n(t_2)]\,\varphi_i(t_1)\varphi_i(t_2)\,dt_1 dt_2$$

$$R_n(t_1-t_2) = \eta_0/2\,\delta(t_1-t_2) \qquad \text{Bianco}$$

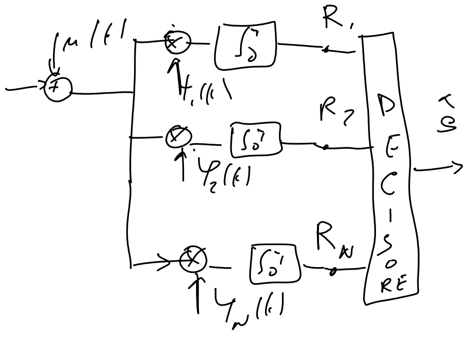
$$= \frac{\eta_0}{2}\underbrace{\int_0^T \varphi_i^2(t)\,dt}_{1} = \eta_0/2 \quad \forall i$$

$$R_x(\tau) = \alpha\,\delta(\tau)$$
$$P_x(f) = \alpha$$

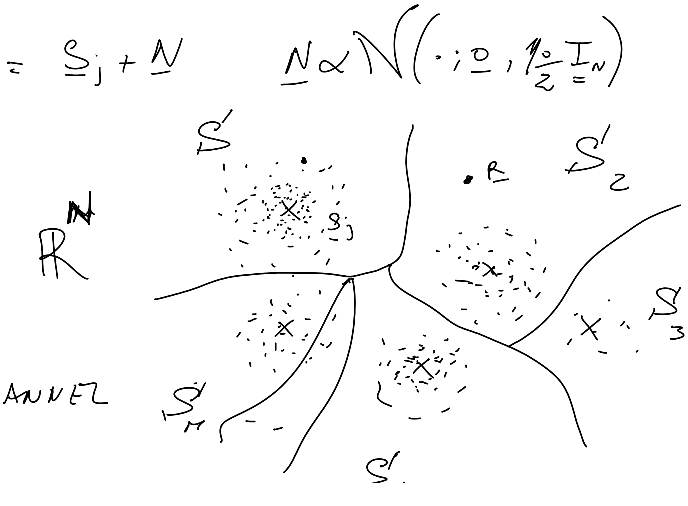


$$E[N_iN_j] = E\left[\int_0^T n(t_1)\varphi_i(t)dt \int_0^T n(t_2)\varphi_j(t)dt\right] = \qquad i \neq j$$

$$= \int_0^T\int_0^T \underbrace{E[n(t_1)n(t_2)]}_{\eta_0/2\,\delta(t_1-t_2)}\varphi_i(t_1)\varphi_j(t_2)\,dt_1 dt_2 = \frac{\eta_0}{2}\underbrace{\int_0^T \varphi_i(t)\varphi_j(t)\,dt}_{0 \quad i\neq j} = 0$$



$$\underline{R}|\alpha_j = \underline{S}_j + \underline{N} \qquad \underline{N} \propto \mathcal{N}(\cdot\,;\underline{0}, \tfrac{\eta_0}{2}\underline{\underline{I}}_N)$$

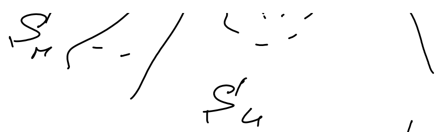


AWGN - CHANNEL

ADDITIVE WHITE GAUSSIAN NOISE - CHANNEL

$R \rightarrow \hat{s}$

ADDITIVE WHITE GAUSSIAN NOISE - CHANNEL



DECISORE : $\underline{R} \to \hat{s}$

BASATO SU PARTIZIONE $S_1, S_2, \dots S_M$: $\bigcup_{j=1}^{M} S_j = \mathbb{R}^N$, $S_i \cap S_j = \emptyset$

TROVARE LA PARTIZIONE OTTIMA

$$P(e) = P_r\{\hat{S} \neq s\} = 1 - P(c) = 1 - P_r\{\hat{S} = s\}$$

$$= 1 - \sum_{j=1}^{M} p(c|s=a_j)\pi_j = 1 - \sum_{j=1}^{M} P_r\{\underline{R} \in S_j | s=a_j\}\pi_j$$

$$= 1 - \sum_{j=1}^{M} \underbrace{\int_{S_j} f_{\underline{R}}(\underline{r}|a_j)\,d\underline{r}\;\pi_j}$$

Le regioni ottime sono quelle che massimizzano questa sommatoria

$$S_j \text{ ottima} = \left\{\underline{r} : \boxed{f_{\underline{R}}(\underline{r}|a_j)\pi_j > f_{\underline{R}}(\underline{r}|a_i)\pi_i} ,\ \forall i \neq j\right\}$$

CRITERIO MAP (MASSIMA PROB. A POSTERIORI)

$$\frac{f_{\underline{R}}(\underline{r}|a_j)\pi_j}{f_{\underline{R}}(\underline{r})} > \frac{f_{\underline{R}}(\underline{r}|a_i)\pi_i}{f_{\underline{R}}(\underline{r})}$$

$$P(a_j|\underline{r}) > p(a_i|\underline{r})$$

PROB. A POSTERIORI

SE $\pi_j = \pi_i = \frac{1}{M}$ Prob. a priori uniformi.

$$f_{\underline{R}}(\underline{r}|a_j) > f_{\underline{R}}(\underline{r}|a_i)$$

MASSIMA VEROSIMIGLIANZA

(ML MAXIMUM LIKELIHOOD)

USIAMO L'IPOTESI GAUSSIANA

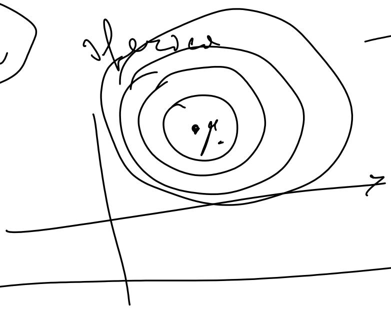
$$f_{\underline{R}}(\underline{r}|a_j) = N\left(\underline{r} ; \underline{S}_j , \frac{\eta_0}{2}\underline{\underline{I}}_N\right) \quad j = 1, \dots M$$

Gaussiana N-dimensionale.

$$N(\underline{r}, \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{r}-\underline{\mu})^T \underline{\underline{\Sigma}}^{-1}(\underline{r}-\underline{\mu})}$$

$N(\underline{r};\underline{\mu}\ \ \ )\quad (2\pi)\ |\Sigma|$

$$\boxed{\text{caso}\ \underline{\underline{\Sigma}} = \sigma^2 \underline{\underline{I}}_N}$$



$$= \frac{1}{(2\pi)^{N/2}\sqrt{\sigma^2}^{N}} e^{-\frac{1}{2\sigma^2}(\underline{r}-\underline{\mu})^T(\underline{r}-\underline{\mu})}$$

$$= \boxed{\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(r_i-\mu_i)^2}}$$

$$f_{\underline{R}}(\underline{r}|a_j)\pi_j > f_{\underline{R}}(\underline{r}|a_i)\pi_i$$

$$N\left(\underline{r} ; \underline{s}_j , \frac{\eta_0}{2}\underline{\underline{I}}_N\right)\pi_j > N\left(\underline{r} ; \underline{s}_i , \frac{\eta_0}{2}\underline{\underline{I}}_N\right)$$

$$\prod_{\ell=1}^{N} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_\ell - s_{j\ell})^2} \pi_j > \prod_{\ell=1}^{N} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_\ell - s_{i\ell})^2} \pi_i$$

Lezione 18

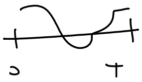
venerdì 4 dicembre 2020 11.01

n(t) bianco $R_n(\tau)=\frac{\eta_0}{2}\delta(\tau)$

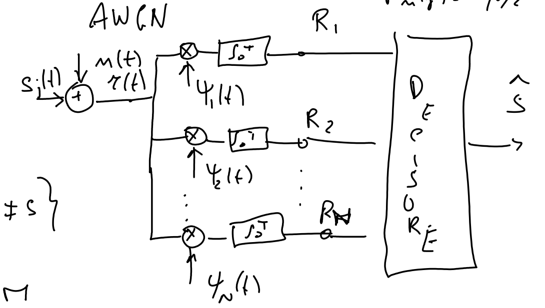
$P_n(f)=\eta_0/2$

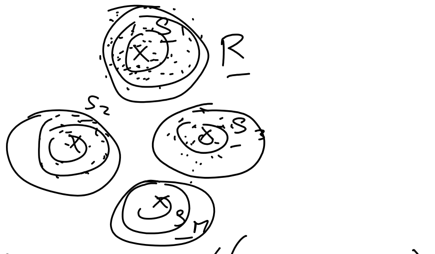
$\{s_1(t)\ s_2(t)\ \dots\ s_M(t)\}$

$a_1\quad a_2\quad a_M$



AWGN





$P(e): \Pr\{\hat{s}\neq s\}$

$N \leq M$

min $P(e) \Longleftrightarrow$ max $P(c) \Longleftrightarrow$ MAP

$$f_{\underline{R}}(\underline{r}|a_j)=\mathcal{N}\left(\underline{r};\underline{s}_j,\frac{\eta_0}{2}\underline{\underline{I}}_N\right)$$

scegli a_j se $f_{\underline{R}}(\underline{r}|a_j)\pi_j > f_{\underline{R}}(\underline{r}|a_i)\pi_i$

$2\pi\frac{\eta_0}{2}$

$$\mathcal{N}\left(\underline{r};\underline{s}_j,\frac{\eta_0}{2}\underline{\underline{I}}_N\right)\pi_j > \mathcal{N}\left(\underline{r};\underline{s}_i,\frac{\eta_0}{2}\underline{\underline{I}}_N\right)\pi_i$$

$$\prod_{\ell=1}^{N}\frac{1}{\sqrt{\pi\eta_0}}e^{-\frac{(r_\ell-s_{j\ell})^2}{\eta_0}}\pi_j > \prod_{\ell=1}^{N}\frac{1}{\sqrt{\pi\eta_0}}e^{-\frac{(r_\ell-s_{i\ell})^2}{\eta_0}}\pi_i$$

$$\cancel{\left(\frac{1}{\sqrt{\pi\eta_0}}\right)^N}e^{-\frac{1}{\eta_0}\sum_{\ell=1}^{N}(r_\ell-s_{j\ell})^2}\pi_j > \cancel{\left(\frac{1}{\sqrt{\pi\eta_0}}\right)^N}e^{-\frac{1}{\eta_0}\sum_{\ell=1}^{N}(r_\ell-s_{i\ell})^2}\pi_i$$

$\Downarrow$ ln

$$-\frac{1}{\eta_0}\sum_{\ell=1}^{N}(r_\ell-s_{j\ell})^2+\ln\pi_j > -\frac{1}{\eta_0}\sum_{\ell=1}^{N}(r_\ell-s_{i\ell})^2+\ln\pi_i$$

$$\mathcal{N}(\underline{x};\underline{\mu},\underline{\underline{\Sigma}})=\frac{1}{(2\pi)^{N/2}|\underline{\underline{\Sigma}}|^{1/2}}e^{-\frac{(\underline{x}-\underline{\mu})^T\underline{\underline{\Sigma}}^{-1}(\underline{x}-\underline{\mu})}{2}}$$

$$\underline{\underline{\Sigma}}=\sigma^2\underline{\underline{I}}_N=\begin{pmatrix}\sigma^2 & & 0\\ & \sigma^2 & \\ 0 & & \ddots\ \sigma^2\end{pmatrix}$$

$$\mathcal{N}(\underline{x};\underline{\mu},\sigma^2 I_N)=\frac{1}{(2\pi)^{N/2}\sigma^N}e^{-\frac{(\underline{x}-\underline{\mu})^T(\underline{x}-\underline{\mu})}{2\sigma^2}}=\prod_{\ell=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x_\ell-\mu_\ell)^2}$$

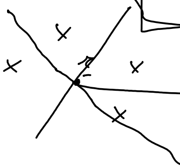
NOTA: se $\pi_i=\frac{1}{M}$ $i:1\dots M$ (ML)

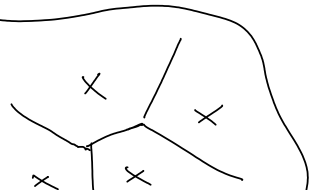
$$-\sum_{\ell=1}^{N}(r_\ell-s_{j\ell})^2 > -\sum_{\ell=1}^{N}(r_\ell-s_{i\ell})^2$$

$$\sum_{\ell=1}^{N}(r_\ell-s_{j\ell})^2 < \sum_{\ell=1}^{N}(r_\ell-s_{i\ell})^2$$

distanza euclidea^2

$\Downarrow$ DECISORE A MINIMA DISTANZA





TESSELLAZIONE DI VORONOI

→ MAP

MV (ML)

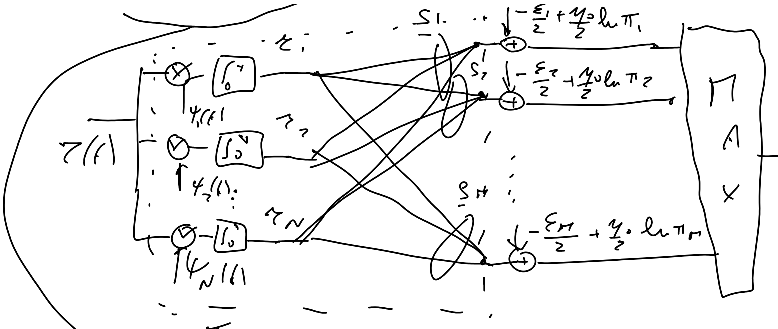
$$-\frac{1}{\eta_0}\sum_{\ell=1}^{N}(\cancel{r_\ell^2}+s_{j\ell}^2-2r_\ell s_{j\ell})+\ln\pi_j > -\frac{1}{\eta_0}\sum_{\ell=1}^{N}(\cancel{r_\ell^2}+s_{i\ell}^2-2r_\ell s_{i\ell})+\ln\pi_i$$

$$\frac{2}{\eta_0}\sum_{\ell=1}^{N}r_\ell s_{j\ell}-\frac{1}{\eta_0}\underbrace{\sum_{\ell=1}^{N}s_{j\ell}^2}_{\mathcal{E}_j}+\ln\pi_j > \frac{2}{\eta_0}\sum_{\ell=1}^{N}r_\ell s_{i\ell}-\frac{1}{\eta_0}\underbrace{\sum_{\ell=1}^{N}s_{i\ell}^2}_{\mathcal{E}_i}+\ln\pi_i$$

$$\sum_{\ell=1}^{N}r_\ell s_{j\ell}-\frac{1}{2}\mathcal{E}_j+\frac{\eta_0}{2}\ln\pi_j > \sum_{\ell=1}^{N}r_\ell s_{i\ell}-\frac{1}{2}\mathcal{E}_i+\frac{\eta_0}{2}\ln\pi_i$$

Spazio dei segnali è ortonormale

$$\int s_j^2(t)dt=\|\underline{s}_j\|^2$$

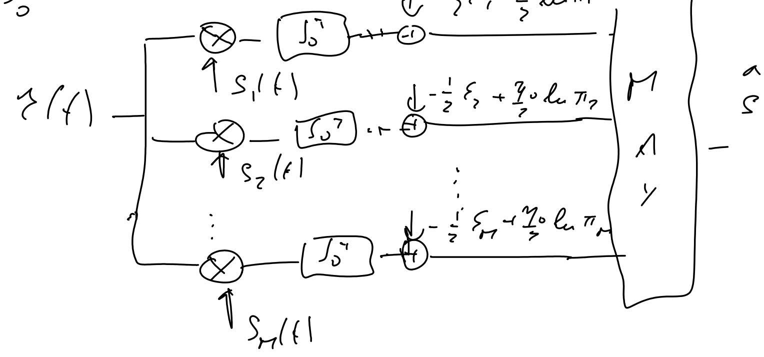


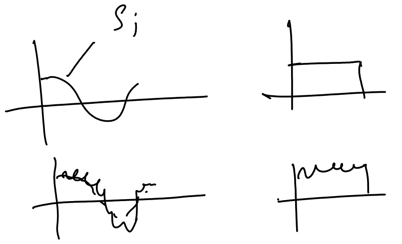
$$\int_0^T s_j^2(t)dt = \|s_j\|^2$$

$$\mathcal{E}_j \longleftrightarrow$$

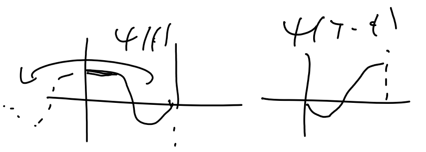
RIC. OTTIMO

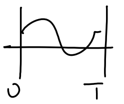
$$\int_0^T r(t)s_j(t)dt$$

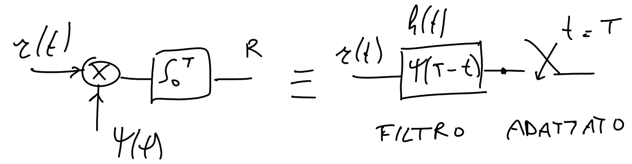




$$\psi(T-t) = \psi(-(t-T))$$



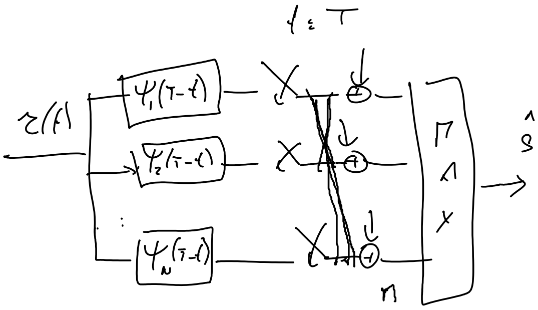


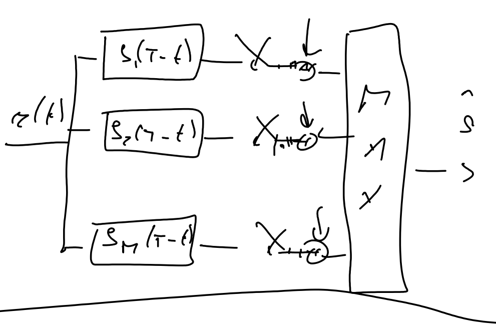


FILTRO ADATTATO

$$\int_0^T r(\tau)\,h(t-\tau)\,d\tau\Big|_{t=T} = \int_0^T r(\tau)\,\psi(T-t+\tau)\,d\tau\Big|_{t=T}$$

$$= \int_0^T r(\tau)\,\psi(\tau)\,d\tau$$

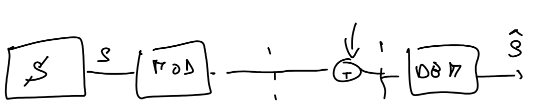


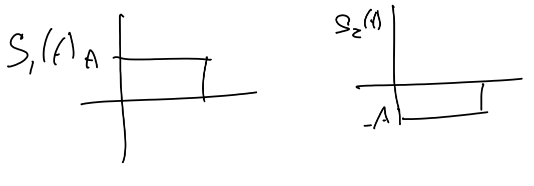


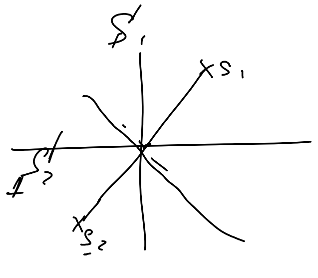
CASO BINARIO

$S \in \{a_1, a_2\}$

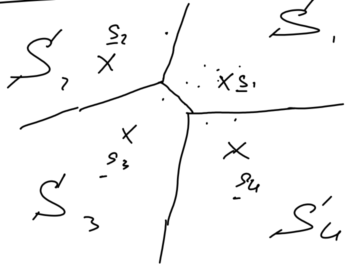
$\pi \in \{p, (1-p)\}$







⇒ casi particoli → ric. semplici

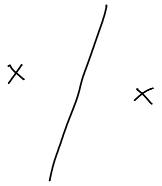


$$P(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} P(c|a_i)\pi_i$$

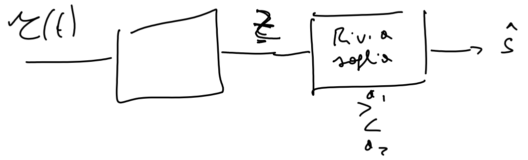
$$= 1 - \sum_{i=1}^{M} \int_{S'_i} f_{\underline{R}}(\underline{z}|a_i)\,d\underline{z}\,\pi_i$$

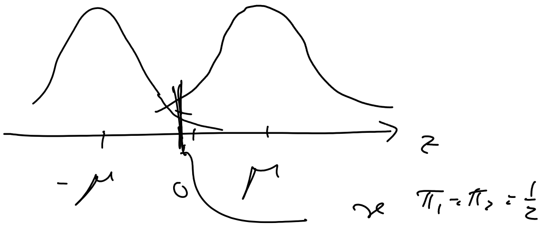
$$= 1 - \sum_{i=1}^{M} \int_{S'_i} N\left(\underline{z}; \underline{s}_i, \frac{\eta_0}{2}\underline{I}_N\right) d\underline{z}\,\pi_i$$

CASO BINARIO



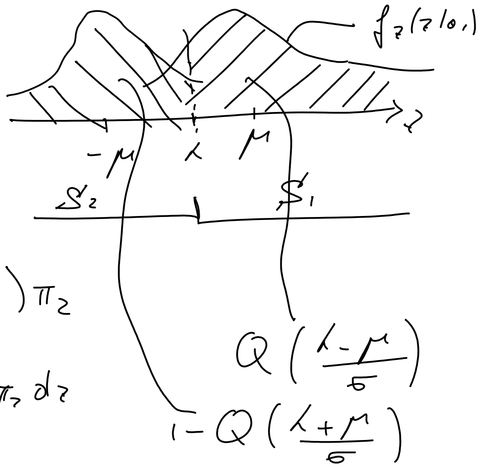
$$\underbrace{\langle \underline{z}, \underline{s}_1 - \underline{s}_2 \rangle - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_2)}_{z} \underset{a_2}{\overset{a_1}{\gtrless}} \underbrace{\frac{\eta_0}{2} \ln \frac{\pi_2}{\pi_1}}_{\text{soglia } \lambda}$$





$$f_z(z|a_1) = N\left(z; \underbrace{\frac{1}{2} d_{12}^2}_{}, \frac{\eta_0}{2} d_{12}^2\right)$$

$$f_z(z|a_2) = N\left(z; -\frac{1}{2} d_{12}^2, \frac{\eta_0}{2} d_{12}^2\right)$$



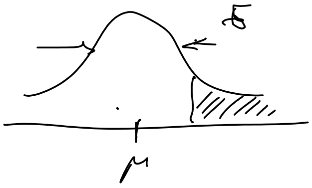
$$P(e) = 1 - P(c) = 1 - P(c|a_1)\pi_1 - P(c|a_2)\pi_2$$

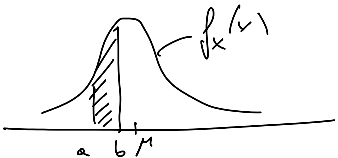
$$= 1 - \int_{S'_1} f_z(z|a_1)\,dz\,\pi_1 - \int_{S'_2} f_z(z|a_2)\,\pi_2\,dz$$

$$= 1 - \int_{\lambda}^{\infty} N(z; \mu, \sigma^2)\,dz\,\pi_1 - \int_{-\infty}^{\lambda} N(z; -\mu, \sigma^2)\,dz\,\pi_2 =$$

$$N(x; \mu, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

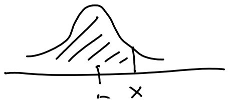




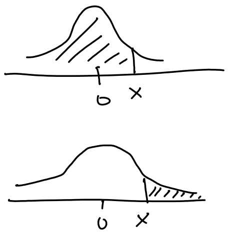
$$P_r\{a < X < b\} = \int_a^b f_x(x)\,dx = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

$$= Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

$N(x, 0, 1)$ funzione normalizzata



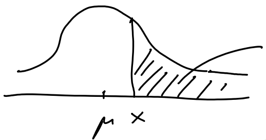
$$\phi(x) \triangleq \int^{x} N(\xi, 0, 1)\,d\xi = 1 - Q(x)$$



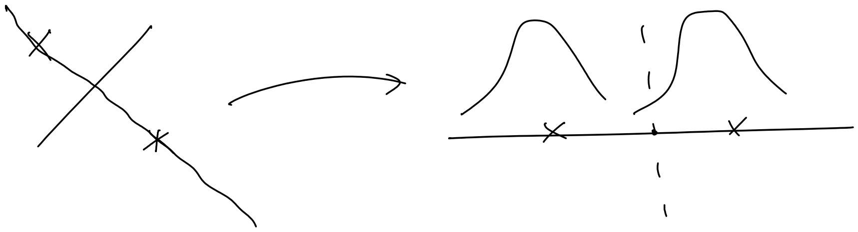
$$\phi(x) \triangleq \int_{-\infty}^{x} N(\xi,0,1)\,d\xi = 1 - Q(x)$$

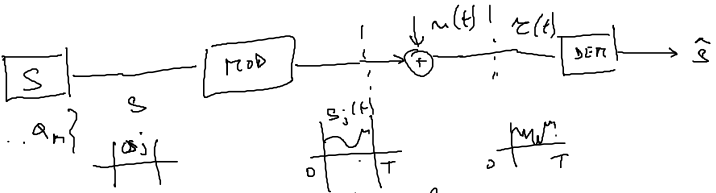
$$Q(x) = \int_{x}^{\infty} N(\xi,0,1)\,d\xi = 1 - \phi(x)$$

$$= Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$



$$Q\left(\frac{x-\mu}{\sigma}\right)$$

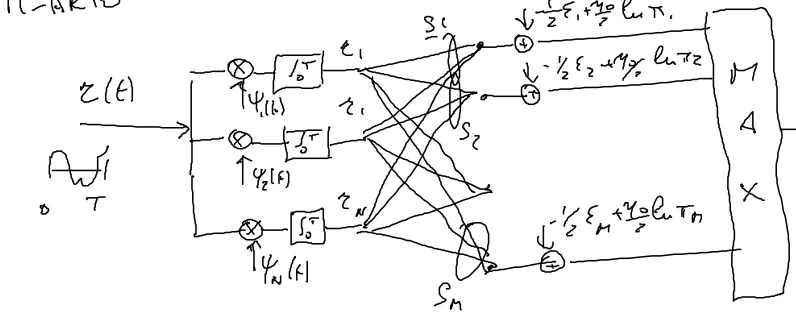


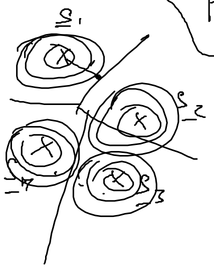


$S \in \{a_1 \dots a_M\}$

M-ARIO

n(t) processo bianco $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$ $P_n(f) = \frac{N_0}{2}$

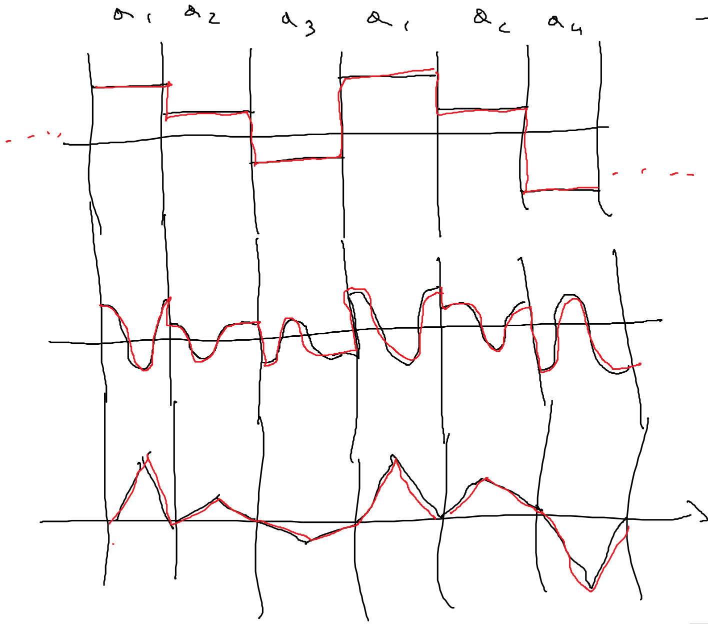


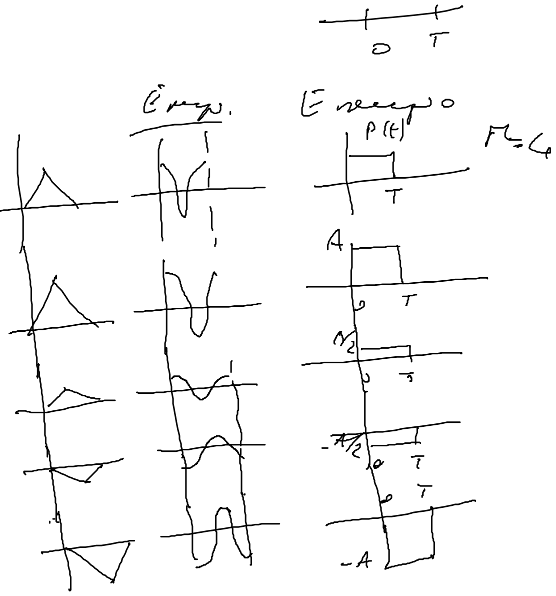


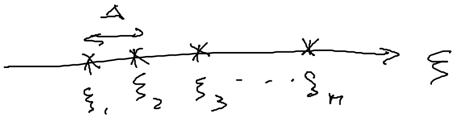
MODULAZIONE PAM (PULSE AMPLITUDE MODULATION)

$\mathcal{A} = \{a_1, \dots, a_M\} \longrightarrow \{s_1(t), s_2(t), \dots s_M(t)\}$

$s_j(t) = \xi_j\, p(t)$

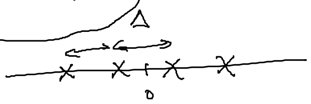






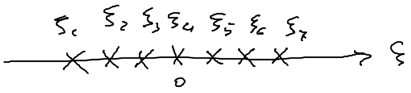
$$\boxed{\xi_j = -\frac{M+1}{2}\Delta + j\Delta \qquad j = 1, \dots, M}$$

M=4



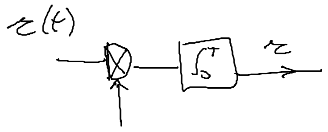
$-\frac{5}{2}\Delta + j\Delta$

M=7

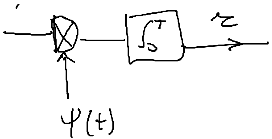


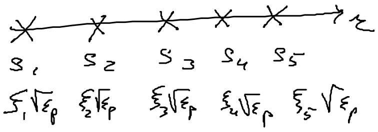
$s_j(t) = \xi_j\, p(t)$

$\psi'(t) = p(t) \quad ; \quad \psi(t) = \frac{\psi'(t)}{\sqrt{\|\psi'\|^2}} = \frac{p(t)}{\sqrt{\mathcal{E}_p}}$



M=5





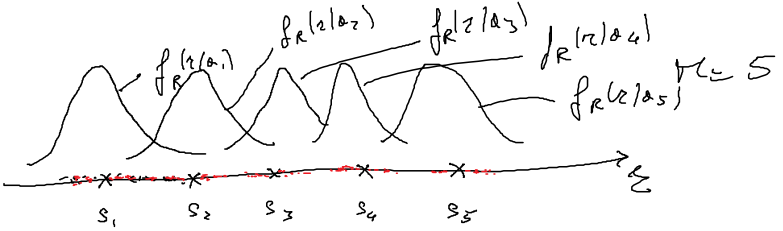
$$s_j = \int_0^T s_j(t)\psi(t)\,dt = \xi_j \int_0^T p(t)\frac{p(t)}{\sqrt{\mathcal{E}_p}}\,dt$$

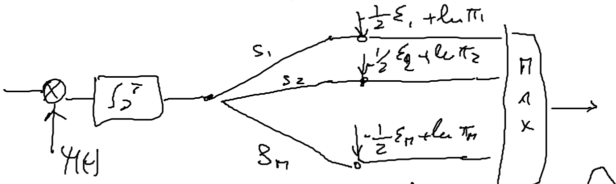
$$= \xi_j\sqrt{\mathcal{E}_p}$$

$$r/a_j = \int_0^T r(t)\psi(t)\,dt = \int_0^T \left(s_j(t)+n(t)\right)\psi(t)\,dt = \underbrace{\xi_j\sqrt{\mathcal{E}_p}}_{s_j} + \underbrace{\int_0^T n(t)\psi(t)\,dt}_{N}$$

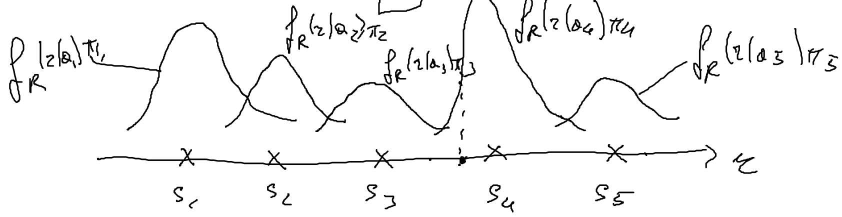
N gaussiana, media nulla, varianza $\eta_0/2$

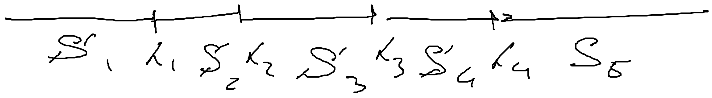
$$f_R(r|a_j) = \mathcal{N}(r; s_j, \eta_0/2) \qquad j = 1 \dots M$$

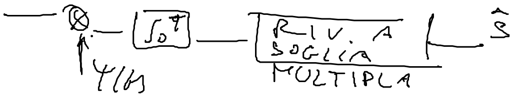




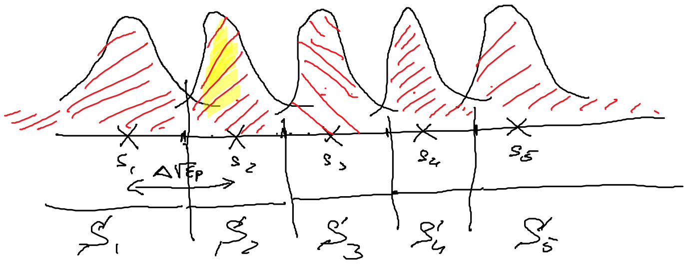
$$f_R(r|a_j)\pi_j > f_R(r|a_i)\pi_i$$







Ric. a M.V. $\pi_j = \frac{1}{M}$ $j = 1..M$ $\qquad f_R(r|a_j) > f_R(r|a_i)$



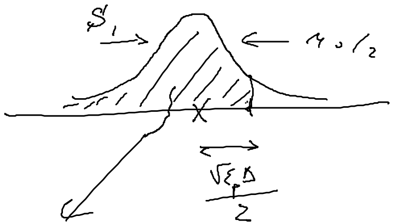
soglie uniformi

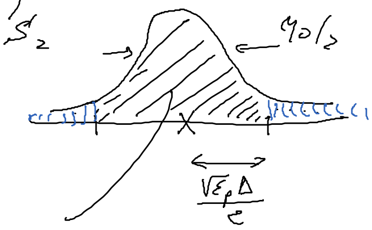
$$P(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} P(c|a_i)\pi_i = 1 - \sum_{i=1}^{M} \int_{S_i} f_R(r|a_i)\,dr\;\pi_i$$

$$= 1 - \sum_{i=1}^{M} \int_{S_i} \mathcal{N}(r; s_i, \eta_0/2)\,dr\;\pi_i$$

$$= 1 - (\pi_1 + \pi_M)\int \mathcal{N}(r; s_1, \eta_0/2) - (\pi_2 + \pi_3 \dots + \pi_{M-1})\int \mathcal{N}(r; s_2, \eta_0/2)\,dr$$

$$= 1 - (\Pi_1 + \Pi_M) \int N(r; s_1, \eta_0/2) - (\Pi_2 + \Pi_3 \dots + \Pi_{M-1}) \int N(r; s_2, \eta_0/2)\, dz$$





$$1 - Q\left(\frac{\sqrt{\mathcal{E}_p}\Delta}{2\sqrt{\frac{\eta_0}{2}}}\right) = 1 - Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)$$

$$1 - 2Q\left(\frac{\sqrt{\mathcal{E}_p}\Delta}{2\sqrt{\frac{\eta_0}{2}}}\right)$$

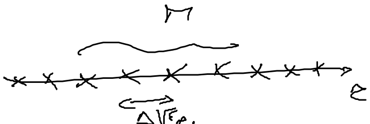
$$1 - 2Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)$$

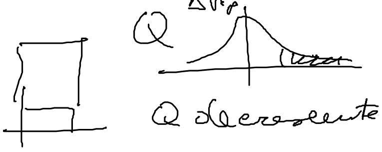
$$P(e) = 1 - (\Pi_1 + \Pi_M)(1 - Q(\cdot)) - (\Pi_2 + \dots + \Pi_{M-1})(1 - 2Q(\cdot))$$

$$\Pi_i = \frac{1}{M}$$

$$= 1 - \frac{2}{M}(1 - Q(\cdot)) - \frac{M-2}{M}(1 - 2Q)$$

$$\boxed{P(e) = 2\left(1 - \frac{1}{M}\right) Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)}$$





Q decrescente

$\mathcal{E}_p$, Δ

M, T

η_0

$\longrightarrow P(e)$

$$B_r = \frac{\lg_2 M}{T}$$

$\lg_2 M$

Energia non è uniforme
⇒ segnali non equienergetici.

$$\frac{\overline{\mathcal{E}_b}}{\eta_0} = \frac{\overline{\text{Energia per bit}}}{\eta_0}$$

$$\mathcal{E}_j = \int_0^T s_j^2(t)\,dt = \xi_j^2 \int_0^T p^2(t)\,dt = \xi_j^2 \mathcal{E}_p$$

$$\xi_j = -\frac{M+1}{2}\Delta + j\Delta$$

$$j = 1 \dots M$$

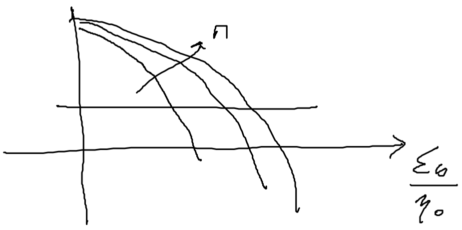
$$\overline{\mathcal{E}} = \frac{1}{M}\sum_{j=1}^{M} \mathcal{E}_j = \frac{\mathcal{E}_p}{M}\sum_{j=1}^{M}\xi_j^2 = \dots = \frac{\mathcal{E}_p \Delta^2 (M^2-1)}{12}$$

$$\frac{\overline{\mathcal{E}_b}}{\eta_0} = \frac{\overline{\mathcal{E}}}{\lg_2 M\, \eta_0} = \frac{\mathcal{E}_p \Delta^2 (M^2-1)}{12 \lg_2 M\, \eta_0}$$

$P(e)$

∘ ∘ ∘ ∘ ∘

$$\boxed{P(e) = 2\left(1 - \frac{1}{M}\right) Q\left(\sqrt{\frac{6\lg_2 M}{M^2-1}\frac{\mathcal{E}_b}{\eta_0}}\right)}$$



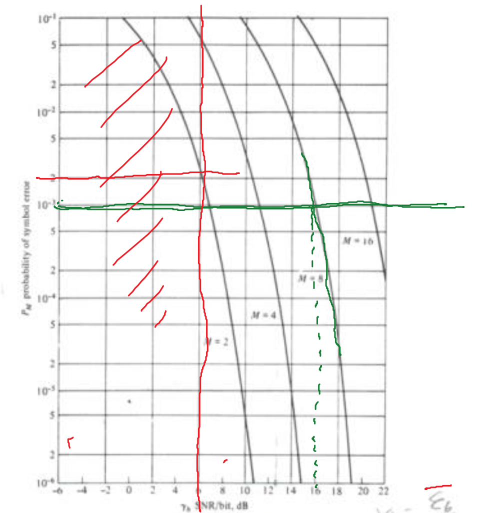


Figure 4.2.24 Probability of a symbol error for PAM.

$\gamma_b = \frac{\mathcal{E}_b}{\eta_0}$

$\boxed{\text{PAM}}$

$M = 8 \quad T = 1\,\mu\text{sec}$

$B_r = \frac{\log_2 M}{T} = \frac{3}{10^{-6}} = 3 \; 10^6$

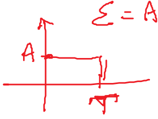
$= 3\,\text{Mbit/sec}$

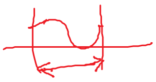
$P(e) < 10^{-3}$

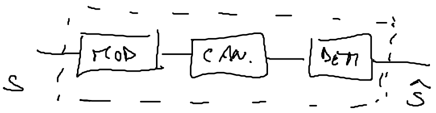
$\frac{\mathcal{E}_b}{\eta_0} > 16\,\text{dB}$

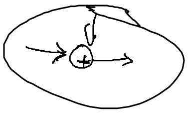
$\mathcal{E}_i = \int_0^T s_i^2(t)\,dt$

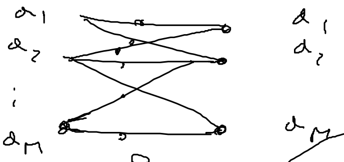
$\mathcal{E} = A^2 T$









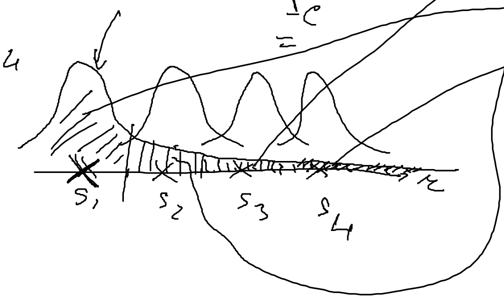


P_c

$=$

M = 4

$\boxed{\text{MV}}$



$P_c = P_r\{\hat{s}=a_1|s=a_1\}P_1\{\hat{s}=a_2|s=a_1\}\; P_r\{\hat{s}=a_3|s=a_1\}P_1\{\hat{s}=a_4|s=a_1\}$

$P_1\{\hat{s}=a_1|s=a_2\}P_2\}$ – – – – –

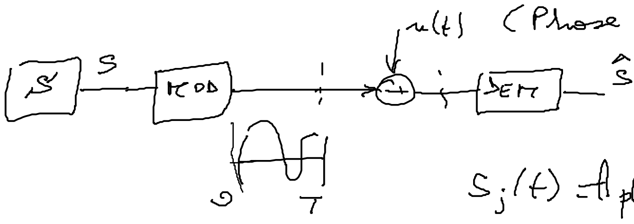
– – – – – –

– – – – – – – .

Appunti al sito dell'anno precedente https://www.mlsptlab-unicampania.it/teaching/trasmissione-ed-elaborazione-numerica-dei-segnali.html#materiale

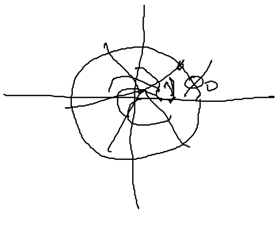
MODULAZIONE PSK

(Phase-Shift Keying)



j = 1, …, M

$$s_j(t) = A\,p(t)\cos(2\pi f_0 t + \theta_j)$$



Base ?

$$\theta_j = \phi_0 + (j-1)\frac{2\pi}{M}$$

$$s_j(t) = A\,p(t)\left(\cos\theta_j \cos 2\pi f_0 t - \sin\theta_j \sin 2\pi f_0 t\right)$$

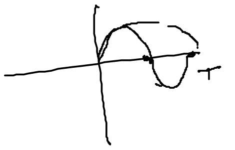
$$\psi_1'(t) = A\,p(t)\cos 2\pi f_0 t$$

$$\psi_2'(t) = -A\,p(t)\sin 2\pi f_0 t$$

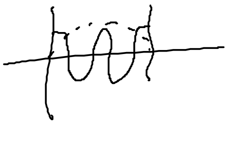
$$\int \psi_1'(t)\psi_2'(t)\,dt = -\int_0^T A^2 p^2(t)\cos 2\pi f_0 t \sin 2\pi f_0 t\,dt$$

$$= A^2\int_0^T \frac{1}{2}\sin 2\pi 2 f_0 t\; p^2(t)\,dt = 0$$

$$f_0 = \frac{K}{T}$$



$$\boxed{f_0 = \frac{K}{T}}$$



$$\langle \psi_1' \psi_1' \rangle = \int_0^T A^2 p^2(t)\overset{\frac{1}{2}+\frac{1}{2}\cos 4\pi f_0 t}{\cos^2 2\pi f_0 t}\,dt$$

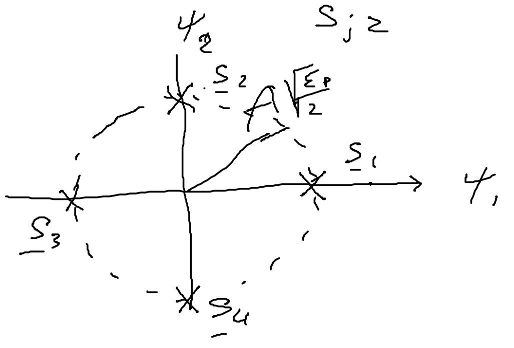
$$= A^2\left[\frac{1}{2}\int_0^T p^2(t)\,dt + \frac{1}{2}\overset{0}{\cancel{\int_0^T p^2(t)\cos 4\pi f_0 t\,dt}}\right] = \frac{A^2}{2}\mathcal{E}_p$$

$$\psi_1(t) = \frac{\sqrt{2}\cancel{A}\,p(t)}{\cancel{A}\sqrt{\mathcal{E}_p}}\cos 2\pi f_0 t = \sqrt{\frac{2}{\mathcal{E}_p}}\,p(t)\cos 2\pi f_0 t$$

$$\psi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_p}}\,p(t)\sin 2\pi f_0 t$$

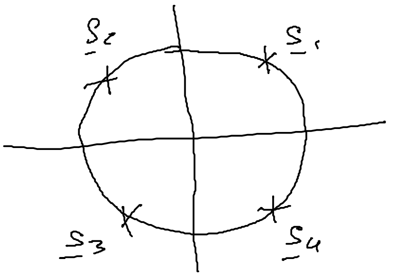
$$s_j(t) = A\sqrt{\frac{\mathcal{E}_p}{2}}\cos\theta_j \underbrace{\sqrt{\frac{2}{\mathcal{E}_p}}\,p(t)\cos 2\pi f_0 t} - A\sqrt{\frac{\mathcal{E}_p}{2}}\sin\theta_j \underbrace{\sqrt{\frac{2}{\mathcal{E}_p}}\,p(t)\sin 2\pi f_0 t}$$

$$= \underbrace{A\sqrt{\frac{\mathcal{E}_p}{2}}\cos\theta_j}_{s_{j1}}\;\psi_1(t) + \underbrace{A\sqrt{\frac{\mathcal{E}_p}{2}}\sin\theta_j}_{s_{j2}}\;\psi_2(t)$$

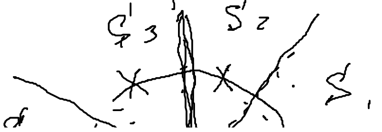


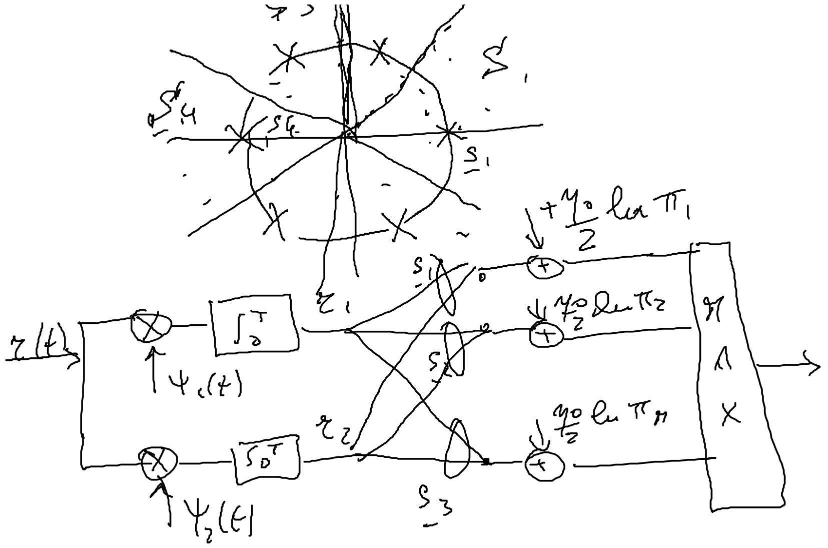
M = 4 , $\phi_0 = 0$

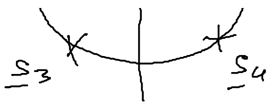
M = 4 , $\phi_0 = \pi/4$



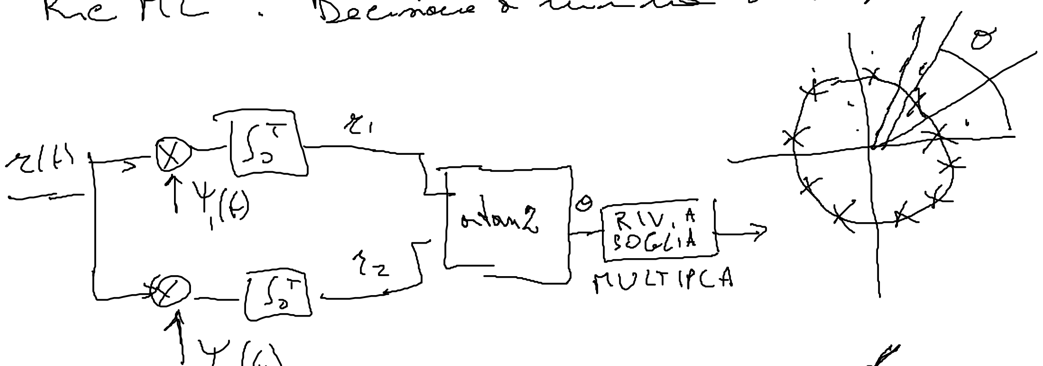
M = 6 $\phi_0 = 0$



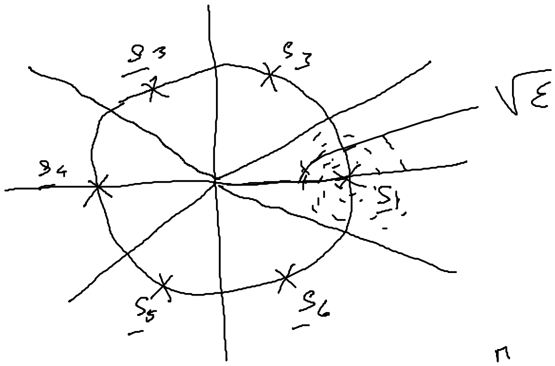




Ric ML : Decisione a minima distanza



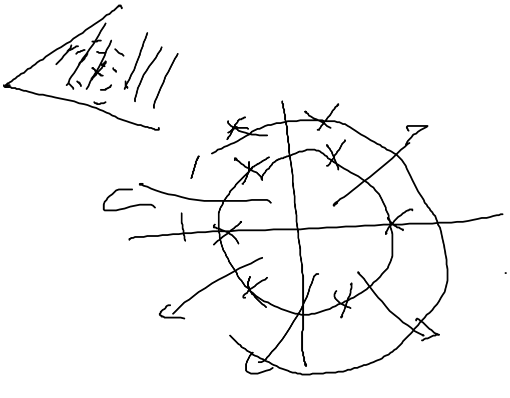
$M = 8$



$\mathcal{N}\left(r; s_i, \frac{N_0}{2} I_2\right)$

$$p(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} p(c|a_i)\pi_i = 1 - \sum_{i=1}^{M} \int_{S_i} f_R(r|a_i)\,dr\,\pi_i$$

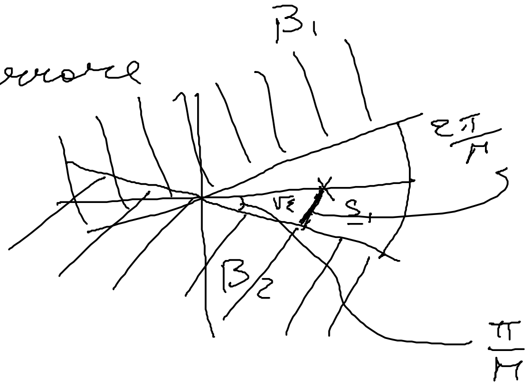
$$= 1 - \int_{S_1} f_R(r|a_1)\,dr \underbrace{\sum_{i=1}^{M}\pi_i}_{1} = 1 - \int_{S_1} \mathcal{N}\left(r; \begin{bmatrix}\sqrt{E}\\ 0\end{bmatrix}, \frac{N_0}{2} I_2\right) dr$$



Maggiori dettagli dagli appunti sul PSK

Appross. della prob. di errore

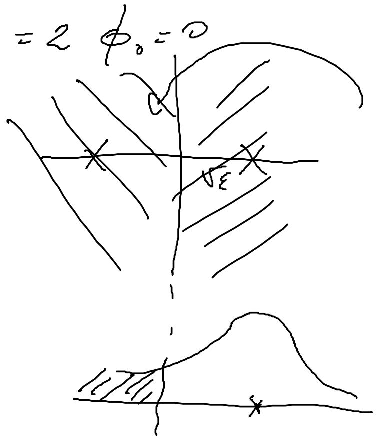
$P(e) < P_r\{r \in B_1 | a_1\} + P_r\{r \in B_2 | a_1\}$



$$Q\left(\frac{\sqrt{E}\sin\frac{\pi}{M}}{\sqrt{\eta_0/2}}\right) = Q\left(\sqrt{\frac{2E}{\eta_0}}\sin\frac{\pi}{M}\right)$$

$$Q(\cdot) < P(e) < 2Q(\downarrow)$$

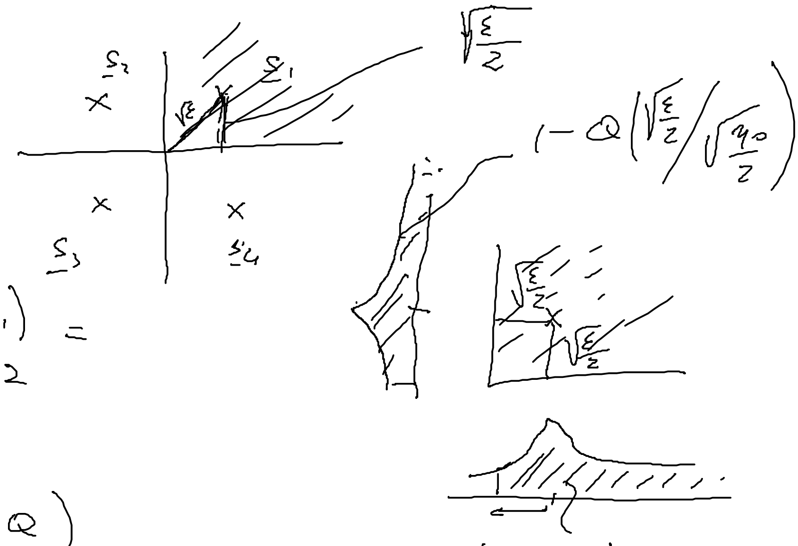
$M = 2 \quad \phi_0 = 0$



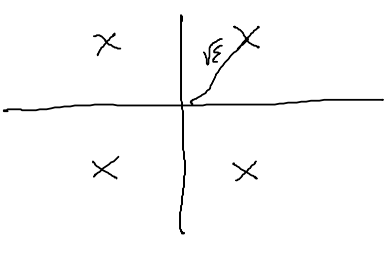
$$P(e) = Q\left(\frac{\sqrt{E}}{\sqrt{\eta_0/2}}\right) = Q\left(\sqrt{\frac{2E}{\eta_0}}\right)$$

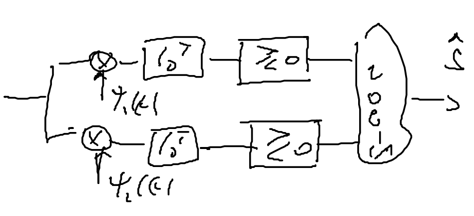
$M = 4 \quad \phi_0 = \pi/4$

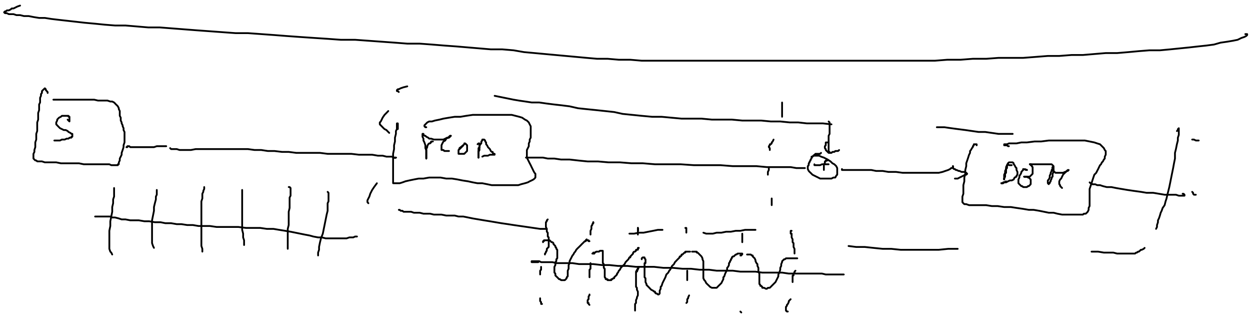
QPSK

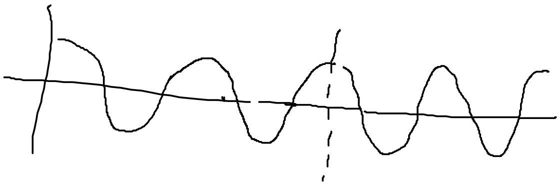


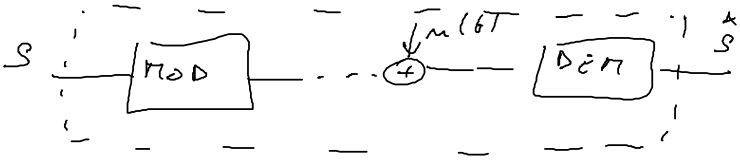
$$P(e) = 1 - P(c) = 1 - P(c|a_1) =$$
$$= 1 - \left(1 - Q\left(\sqrt{\frac{E}{\eta_0}}\right)\right)^2$$
$$= 1 - \left(1 + Q^2(\cdot) - 2Q\right)$$
$$= 2Q(\cdot) - Q^2(\cdot) = 2Q(\cdot)\left(1 - Q(\cdot)\right) \quad 1 - Q\left(\sqrt{\frac{E}{2}}\Big/\sqrt{\frac{\eta_0}{2}}\right)$$

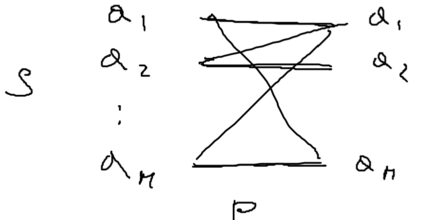




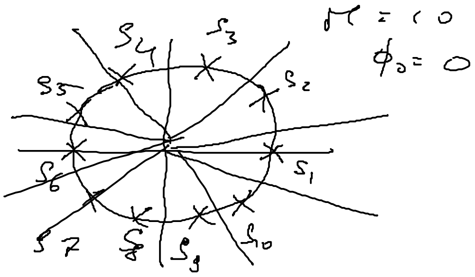






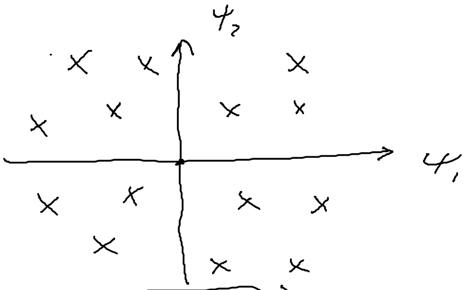


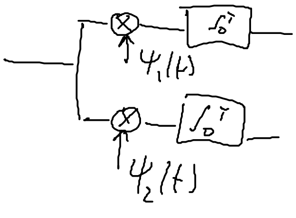
$\underline{P_c}$

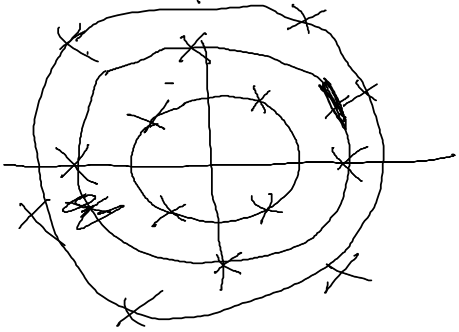


QAM

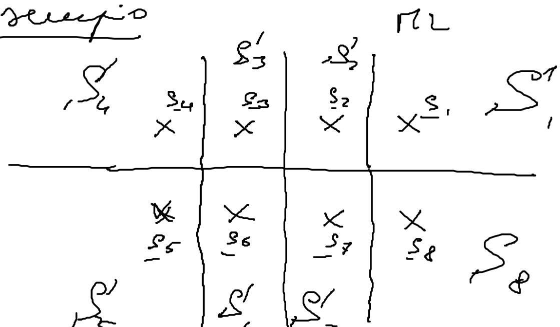
QUADRATURE
AMPLITUDE
MODULATION

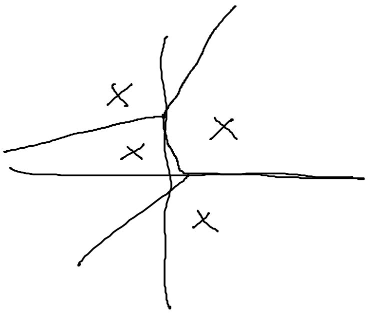


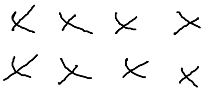


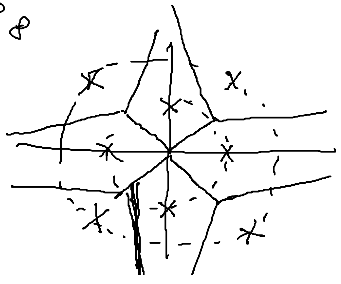


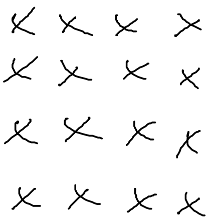
Esempio

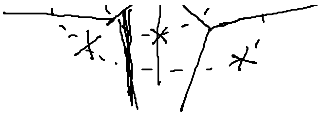


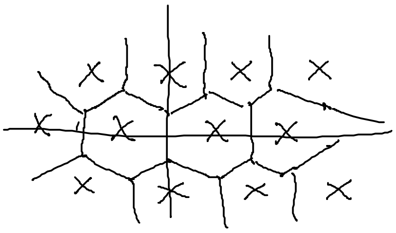


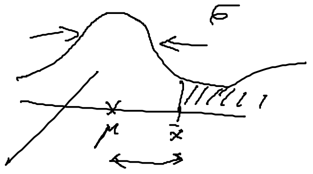












$$Q\left(\frac{\bar{x}-\mu}{\sigma}\right)$$

$$1-Q\left(\frac{\bar{x}-\mu}{\sigma}\right)$$

$$Q\left(\frac{d}{\sqrt{N_0/2}}\right)$$

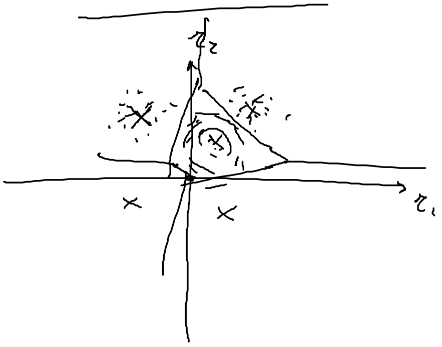
Lezione 21

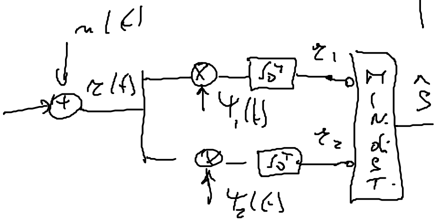
lunedì 14 dicembre 2020 11.01

MOD. QAM

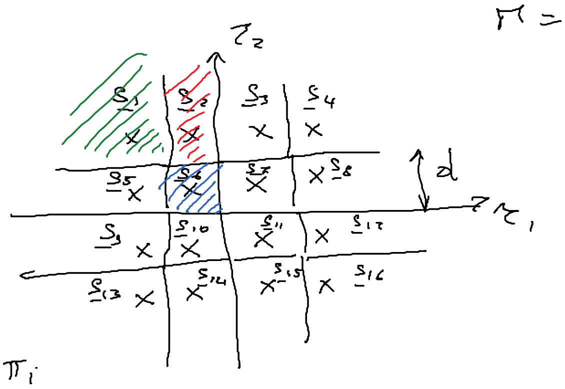
$$s_j(t) = s_{j1}\psi_1(t) + s_{j2}\psi_2(t)$$

$$\underline{s}_j = \begin{bmatrix} s_{j1} \\ s_{j2} \end{bmatrix}$$





M = 16
4 bit.

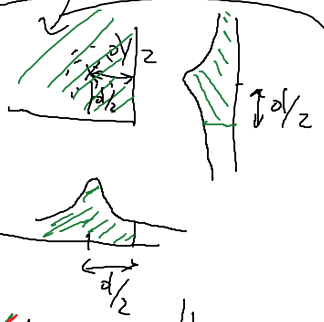


$$P(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} P(c|a_i)\pi_i$$

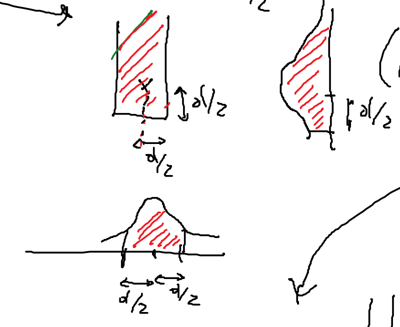
$$= 1 - \sum_{i=1}^{M} \int_{S_i} f_R(\underline{r}|a_i)\,d\underline{r}\,\pi_i = 1 - \sum_{i=1}^{16} \int_{S_i} N(\underline{r}; \underline{s}_i, \eta_0/2\, \underline{I}_2)\,d\underline{r}\,\pi_i$$

$$= 1 - \int_{S_1} N(\underline{r}; \underline{s}_1, \eta_0/2\, I_2)\,d\underline{r}\,(\pi_1 + \pi_4 + \pi_{13} + \pi_{16})$$

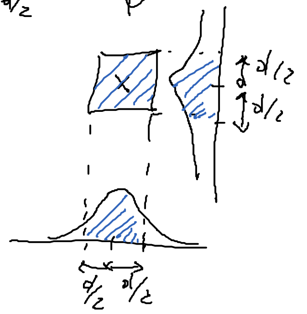
$$- \int_{S_2} N(\underline{r}; \underline{s}_2; \eta_0/2\, \underline{I}_2)\,d\underline{r}\,(\pi_2+\pi_3+\pi_5+\pi_8+\pi_9+\pi_{12}+\pi_{14}+\pi_{15}) - \int_{S_6} N(\underline{r}; \underline{s}_6, \eta_0/2\, \underline{I}_2)\,d\underline{r}\,(\pi_6+\pi_7+\pi_{10}+\pi_{11})$$



$$\left(1 - Q\left(\frac{d/2}{\sqrt{\eta_0/2}}\right)\right)^2$$



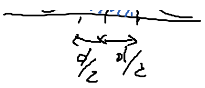
$$\left(1 - Q\left(\frac{d/2}{\sqrt{\eta_0/2}}\right)\right)\left(1 - 2Q\left(\frac{d/2}{\sqrt{\eta_0/2}}\right)\right)$$



$$\left(1 - 2Q\left(\frac{d/2}{\sqrt{\eta_0/2}}\right)\right)^2$$

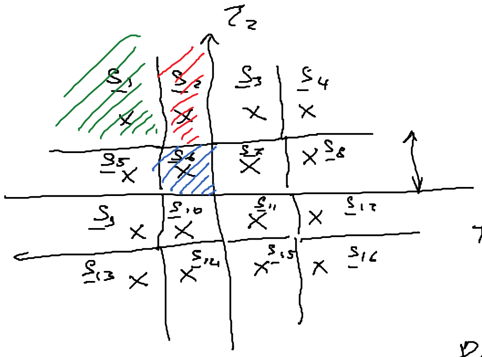
$$\frac{d\sqrt{2}}{2\sqrt{\eta_0}} = \frac{d}{\sqrt{2\eta_0}}$$

1/16



$2\sqrt{\eta_0}$ $\sqrt{2}\eta_0$

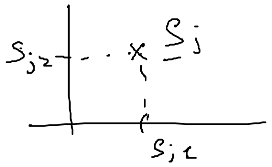
1/16



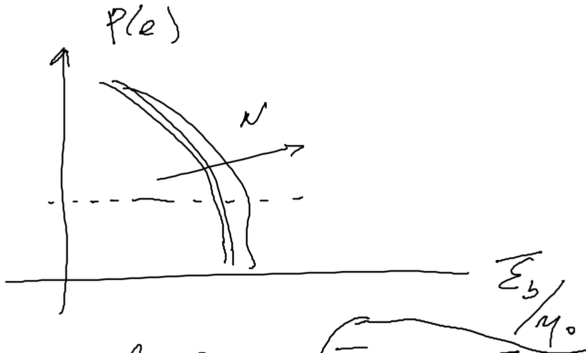
$$P(e) = 1-(\pi_1+\pi_4+\pi_{13}+\pi_{16})\left(1-Q(\cdot)\right)^2$$
$$-(\pi_2+\pi_3+\pi_5+\pi_8+\pi_9+\pi_{12}+\pi_{14}+\pi_5)(1-Q)(1-2Q)$$
$$-(\pi_6+\pi_7+\pi_{10}+\pi_{11})(1-2Q)^2$$

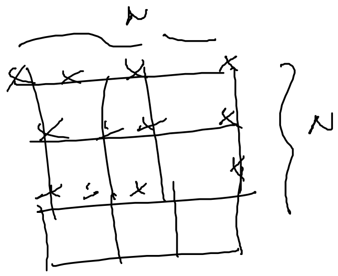
$\pi_i = 1/16$

$$= 1-\frac{1}{4}(1-Q(\cdot))^2-\frac{1}{2}(1-Q)(1-2Q)-\frac{1}{4}(1-2Q)^2$$



$\mathcal{E}_j = s_{j1}^2 + s_{j2}^2$





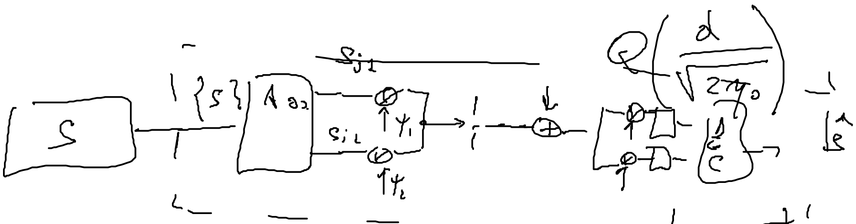
$$\overline{\mathcal{E}} = \frac{d^2(N^2-1)}{6}$$

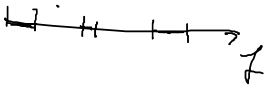
$\log_2 N^2$

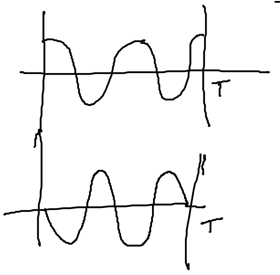
$$\boxed{\frac{\overline{\mathcal{E}}_b}{\eta_0} = \frac{\overline{\mathcal{E}}}{\log_2 N^2 \eta_0} = \frac{d^2(N^2-1)}{12\log_2 N\ \eta_0}}$$

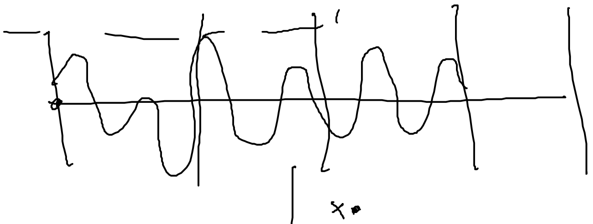
↓

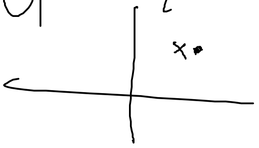
d

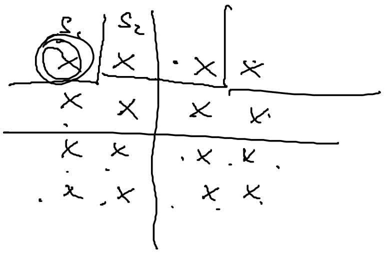


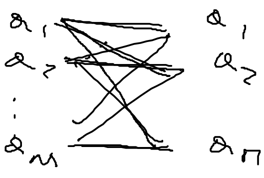


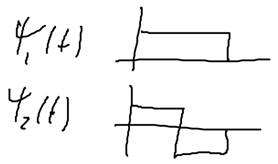












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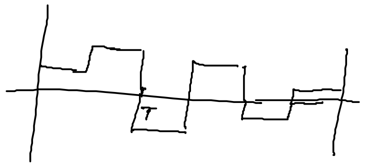
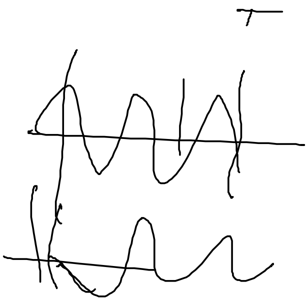
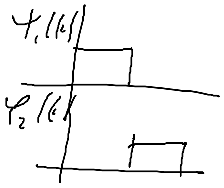
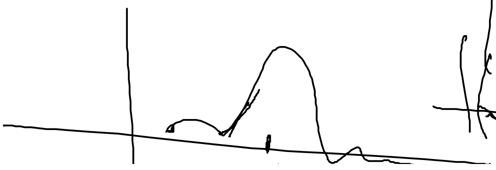
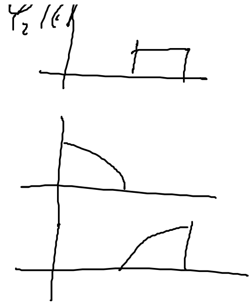
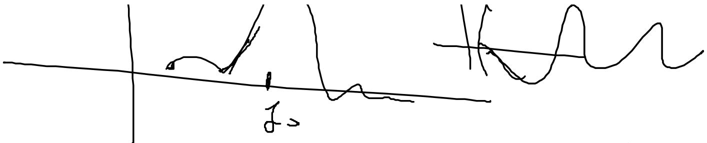


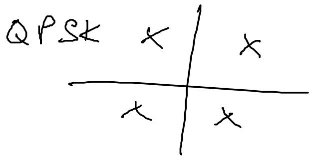








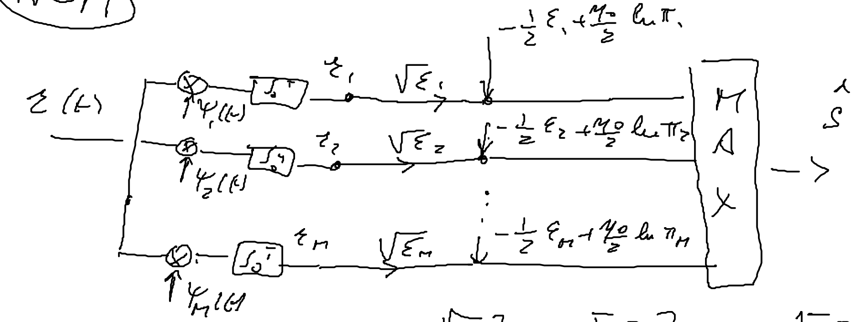


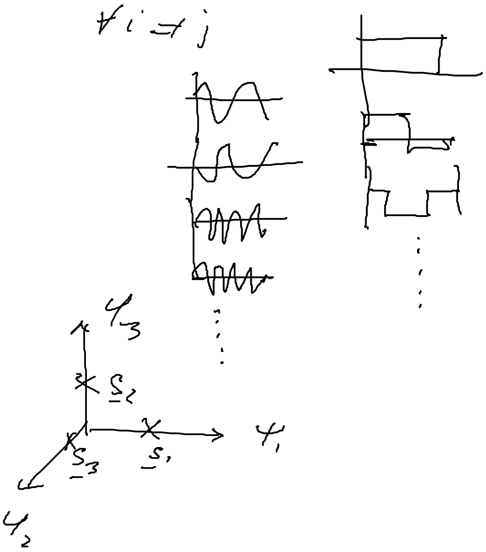


$$S \in \mathcal{A} = \{a_1 \dots a_M\} \longrightarrow \{s_1(t), s_2(t), \dots, s_M(t)\}$$

$$\psi_i(t) = \frac{s_i(t)}{\sqrt{\mathcal{E}_i}} \quad i = 1,\dots,M \qquad s_i \perp s_j \quad \forall i \neq j$$

$\boxed{N=M}$



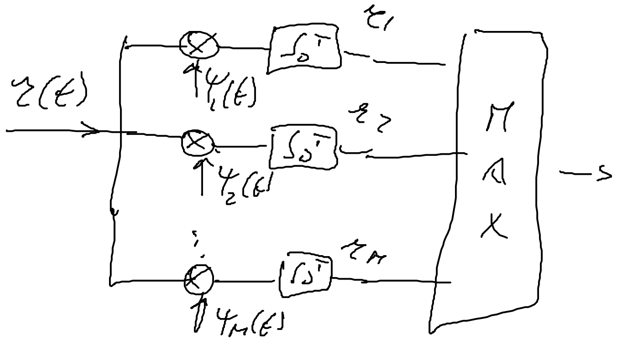


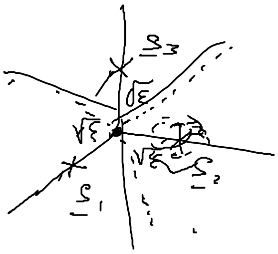
$$\int_0^T s_1(t)\psi_1(t)\,dt = \int_0^T \frac{s_1^2(t)}{\sqrt{\mathcal{E}_1}}\,dt = \sqrt{\mathcal{E}_1}$$

$$\underline{s}_1 = \begin{bmatrix} \sqrt{\mathcal{E}_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} 0 \\ \sqrt{\mathcal{E}_2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \dots \underline{s}_M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \sqrt{\mathcal{E}_M} \end{bmatrix}$$

Caso particolare: $\mathcal{E}_i = \mathcal{E}$ (equienergetici); Ric ML

$M = 3$





$$P(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} p(e|a_i)\pi_i = 1 - \sum_{i=1}^{M} \int f_R(\underline{r}|\underline{s}_i, \tfrac{N_0}{2}\underline{I})\,d\underline{r}\,\pi_i$$

$$= 1 - \int_{S'_1} \mathcal{N}(\underline{r}; \underline{s}_1, \tfrac{N_0}{2}\underline{I})\,d\underline{r} = 1 - P(c|a_1)$$

$M = 3$

$$= 1 - P_r\{r_1 > r_2,\ r_1 > r_3 \,|\, a_1\} = 1 - \int_{S'_1} \mathcal{N}\left(\underline{r}; \begin{bmatrix} \sqrt{\mathcal{E}} \\ 0 \\ 0 \end{bmatrix}, \frac{N_0}{2}\underline{I}_3\right) d\underline{r}$$

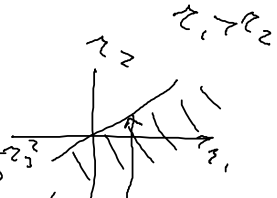
$$S'_1 = \{\underline{r} : r_1 > r_2, r_1 > r_3\}$$

$$= 1 - \int_{-\infty}^{+\infty}\int_{-\infty}^{r_1}\int_{-\infty}^{r_1} \mathcal{N}\left(\underline{r}; \begin{bmatrix} \sqrt{\mathcal{E}} \\ 0 \\ 0 \end{bmatrix}, \frac{N_0}{2}\underline{I}_3\right) d\underline{r}$$

$r_1 > r_2$

r_-

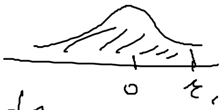
$$= 1 - \int_{-\infty}\int_{-\infty}\int_{-\infty} N\left(\underline{r}; \begin{bmatrix}\sqrt{E}\\0\\0\end{bmatrix}, \frac{\eta_0}{2}\underline{I}_3\right) d\underline{r}$$



$$= 1 - \int_{-\infty}^{+\infty}\int_{-\infty}^{r_1}\int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_1-\sqrt{E})^2} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}r_2^2} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}r_3^2} dr_3\, dr_2\, dr_1$$

$$= 1 - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_1-\sqrt{E})^2} \underbrace{\int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}r_2^2} dr_2}_{} \int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}r_3^2} dr_3 \quad dr_1$$

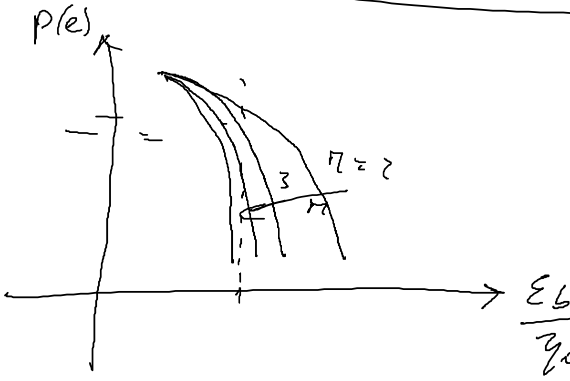
$$= 1 - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_1-\sqrt{E})^2} \left(1 - Q\left(\frac{r_1}{\sqrt{\eta_0/2}}\right)\right)^2 dr_1$$

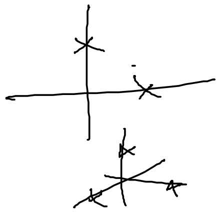


Generalizzando

$$\boxed{P(e) = 1 - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi\eta_0}} e^{-\frac{1}{\eta_0}(r_1-\sqrt{E})^2} \left(1 - Q\left(\frac{r_1}{\sqrt{\eta_0/2}}\right)\right)^{M-1} dr_1}$$

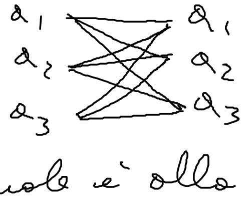
$$E_b = \frac{E}{\log_2 M}$$

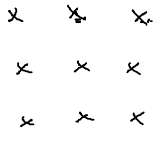




$s_1(t), s_2(t), s_3(t), s_4(t)$

$M \uparrow$ cresce la richiesta di banda.





$$\begin{pmatrix} P_c & \frac{1-P_c}{2} & \frac{1-P_c}{2} \\ \frac{1-P_c}{2} & P_c & \frac{1-P_c}{2} \\ \frac{1-P_c}{2} & \frac{1-P_c}{2} & P_c \end{pmatrix}$$

1

Ogni segnale è alla stessa distanza dagli altri

$$\begin{pmatrix} P_c & \frac{1-P_c}{M-1} & \dots & \frac{1-P_c}{M-1} \\ \frac{1-P_c}{M-1} & P_c & \dots & \frac{1-P_c}{M-1} \\ & & \ddots & \\ \frac{1-P_c}{M-1} & \dots & \frac{1-P_c}{M-1} & P_c \end{pmatrix}$$

CANALE UNIFORME

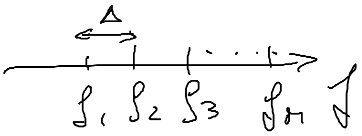
$$\frac{1-P_c}{M-1} \quad \dots \quad \frac{1-P_c}{M-1} \quad P_c$$

MODULAZIONE DI FREQUENZA ORTOGONALE

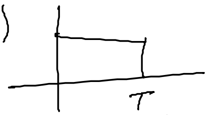
$$s_j(t) = A p(t) \cos 2\pi f_i t$$

$$f_i = f_1 + (j-1)\Delta \qquad j = 1, \dots M$$

$s_j(t)$ ortogonali e equienergetiche



$p(t)$



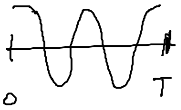
$$s_j(t) = A \cos 2\pi f_i t \qquad t \in [0, T]$$

$$\mathcal{E}_j = \int_0^T s_i^2(t)\,dt = \int_0^T A^2 \cos^2 2\pi f_i t\,dt = A^2 \int_0^T \left(\frac{1}{2} + \frac{1}{2}\cos 2\pi 2 f_i t\right) dt$$

$$= \frac{A^2}{2} T + \frac{A^2}{2} \left. \frac{\sin 2\pi 2 f_i t}{2\pi 2 f_i} \right]_0^T = \frac{A^2}{2} T + \frac{A^2}{2} \frac{\sin 4\pi f_i T}{4\pi f_i}$$

EQUIENERGETICITA': $4\cancel{\pi} f_i T = k_i \cancel{\pi}$

$$f_i = \frac{k_i}{4T}$$

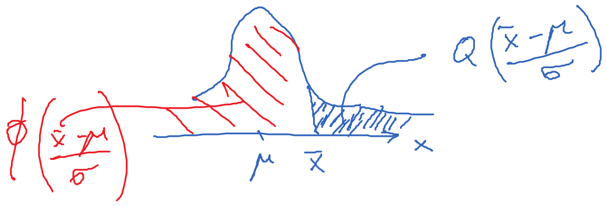


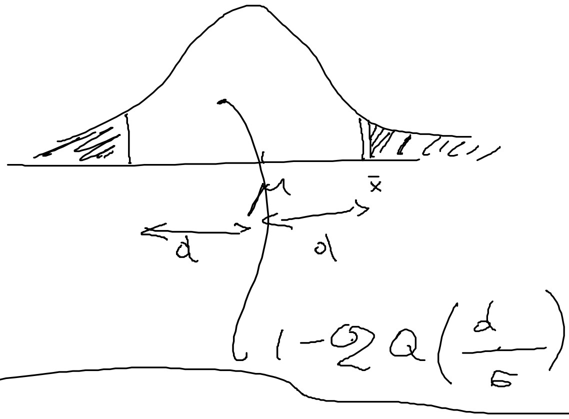
ORTOGONALITA': $\int_0^T s_j(t) s_i(t)\,dt \overset{?}{=} 0$

Lezione 22

venerdì 18 dicembre 2020 11.01

Recap: integrazione della Gaussiana



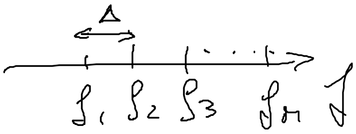


MOD. ORTOGONALE: FSK

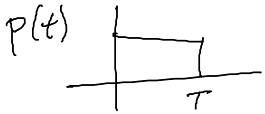
MODULAZIONE DI FREQUENZA ORTOGONALE

$$s_j(t) = A p(t) \cos 2\pi f_i t$$

$$f_i = f_1 + (j-1)\Delta \qquad j = 1, \dots, M$$



$s_j(t)$ ortogonali e equienergetici



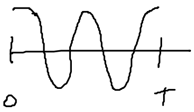
$$s_j(t) = A\cos 2\pi f_i t \qquad t \in [0,T]$$

$$\mathcal{E}_j = \int_0^T s_j^2(t)\,dt = \int_0^T A^2 \cos^2 2\pi f_i t\, dt = A^2 \int_0^T \left(\frac{1}{2} + \frac{1}{2}\cos 2\pi 2 f_i t\right) dt$$

$$= \frac{A^2}{2}T + \frac{A^2}{2} \left.\frac{\sin 2\pi 2 f_i t}{2\pi 2 f_i}\right]_0^T = \frac{A^2}{2}T + \frac{A^2}{2}\frac{\sin 4\pi f_i T}{4\pi f_i}$$

EQUIENERGETICITA': $4\cancel{\pi} f_i T = k_i \cancel{\pi}$

$$f_i = \frac{k_i}{4T}$$



ORTOGONALITA': $\int_0^T s_j(t)\, s_i(t)\, dt \overset{?}{=} 0$

$$\int_0^T \underset{i \neq j}{s_i(t)\, s_j(t)\, dt} = \int_0^T A\cos 2\pi f_i t\; A\cos 2\pi f_j t\, dt$$

$$= \frac{A^2}{2}\int_0^T \cos 2\pi (f_i + f_j)t\, dt + \frac{A^2}{2}\int_0^T \cos 2\pi (f_i - f_j)t\, dt$$

$$= \frac{A^2}{2} \left.\frac{\sin 2\pi (f_i + f_j)t}{f_i + f_j}\right]_0^T + \frac{A^2}{2} \left.\frac{\sin 2\pi (f_i - f_j)t}{f_i - f_j}\right]_0^T$$

$$= \frac{A^2}{2}\left(\frac{\sin 2\pi(f_i+f_j)T}{f_i+f_j} + \frac{A^2}{2}\,\frac{\sin 2\pi(f_i-f_j)T}{f_i-f_j}\right) = 0$$

$= 0$ se $2\cancel{\pi}(f_i+f_j)T = K_2\cancel{\pi}$

$$f_i+f_j = \frac{K_2}{2T}$$

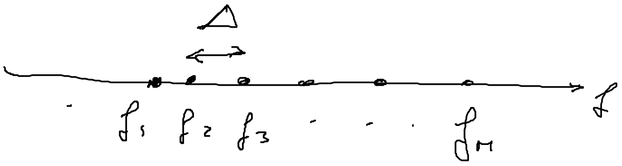
se $2\cancel{\pi}(f_i-f_j)T = K_3\cancel{\pi}$

$$(f_i-f_j) = \frac{K_3}{2T}$$

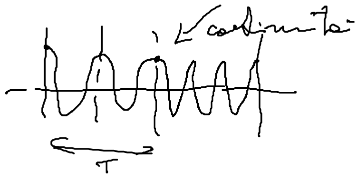
Equienergeticità $\frac{K_1}{4T}$

ortogonalità $f_i+f_j = \frac{K_2}{2T}$

$f_i-f_j = \frac{K_3}{2T}$



$\Delta = \frac{K_3}{2T}$ $\Delta_{min} = \frac{1}{2T}$



$f_i = \frac{K}{T}$

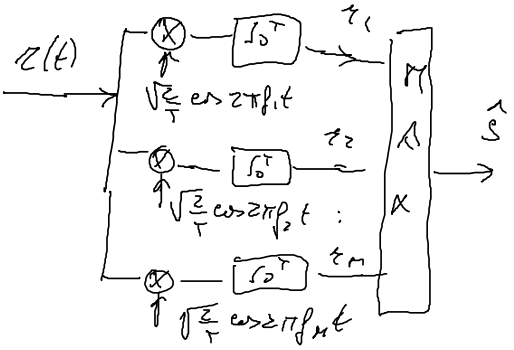
Ric ML $\boxed{f_i = \frac{K}{T}}$ $\Rightarrow$

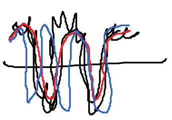
$\varphi_i'(t) = A\cos 2\pi f_i t$

$E_i = \frac{A^2T}{2}$

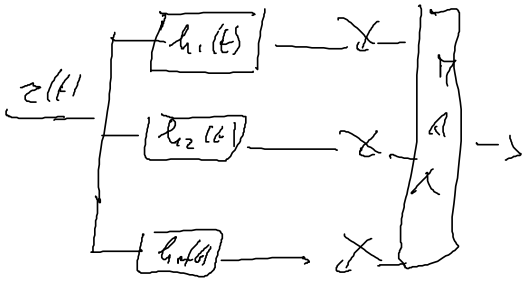
$\varphi_i(t) = \frac{A\sqrt{2}}{A\sqrt{T}}\cos 2\pi f_i t$

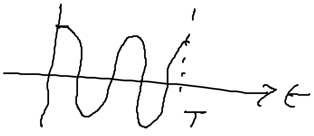
$\varphi_i(t) = \sqrt{\frac{2}{T}}\cos 2\pi f_i t$



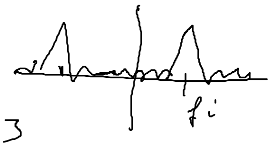


$h_i(t) = \sqrt{\frac{2}{T}}\varphi_i(T-t)$





$H_i(f) =$



M pari

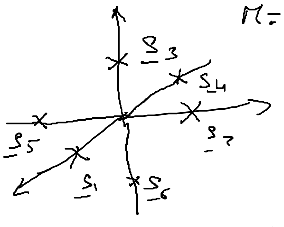
$\{a_1\ a_2\ ..\ a_M\}$

$\downarrow$

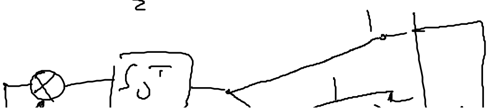
$\{s_1(t)\ ..,s_{\frac{M}{2}}(t), -s_1(t), \ldots, -s_{\frac{M}{2}}(t)\}$

$s_i(t) \perp s_j(t)$

$M = 3$



$\varphi_i(t) = \frac{s_i(t)}{\sqrt{E_i}}$ $i = 1,\ldots,\frac{M}{2}$

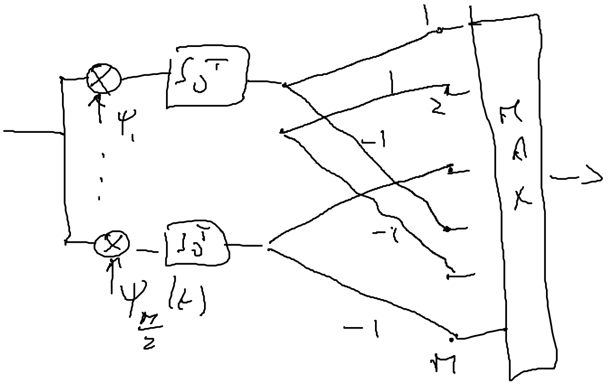


$\sqrt{E_i}$ 2

$s_i(t) \perp s_j(t)$

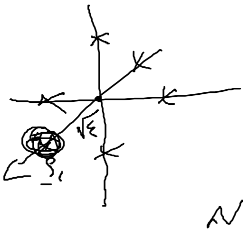
$i \neq j$

$i,j = 1, \dots \frac{M}{2}$



Decidi per a_j se $|R_j| > |R_i|$ e $R_j > 0$ $j,i = 1, \dots, \frac{M}{2}$

" " $a_{j+\frac{M}{2}}$ se $|R_j| > |R_i|$ e $R_j < 0$ "



$N = 3$

$M = 6$

$$P(e) = 1 - P(c) = 1 - \sum_{i=1}^{M} p(c|a_i)\pi_i = 1 - p(c|a_1)$$

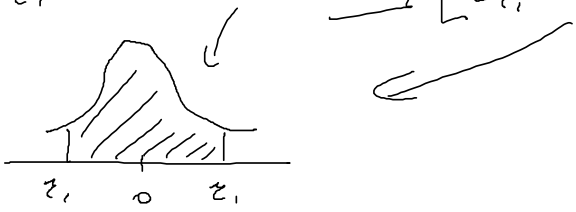
$$p(c|a_1) = P_r\left\{R_1 > 0 \quad |R_1| > |R_2|, |R_1| > |R_3| \,\middle|\, a_1\right\}$$

$$= \int_0^\infty \left[\int_{-r_1}^{r_1}\int_{-r_1}^{r_1} N\left(\underline{r}; \begin{bmatrix}\sqrt{E}\\0\\0\end{bmatrix}, \frac{N_0}{2}\underline{\underline{I}}_3\right) dr_2\, dr_3\right] dr_1$$

$$= \int_0^\infty \int_{-r_1}^{r_1}\int_{-r_1}^{r_1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(r_1-\sqrt{E})^2} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}r_2^2} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}r_3^2}\, dr_2\, dr_3\, dr_1$$

$\sigma^2 = \frac{N_0}{2}$

$$= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_1-\sqrt{E})^2}{N_0}} \left[\int_{-r_1}^{r_1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}r_2^2}\, dr_2\right]\left[\int_{-r_1}^{r_1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}r_3^2}\, dr_3\right] dr_1$$



$$1 - 2Q\left(\frac{r_1}{\sqrt{N_0/2}}\right)$$

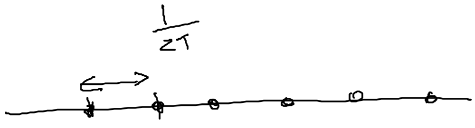
$$= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_1-\sqrt{E})^2}{N_0}} \left(1 - 2Q\left(\frac{r_1}{\sqrt{N_0/2}}\right)\right)^2 dr_1$$

$$P(e) = 1 - \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_1-\sqrt{E})^2}{N_0}} \left(1 - Q\left(\frac{r_1}{\sqrt{N_0/2}}\right)\right)^{M-1} dr_1$$

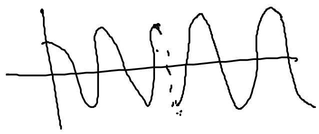
FSK ortogonale FSB - biortogonale

$s_i(t) = A\cos 2\pi f_i t \quad i = 1 .. \frac{M}{2}$

$s_i(t) = -A\cos 2\pi f_i t \quad i = \frac{M}{2}+1, ..., M$

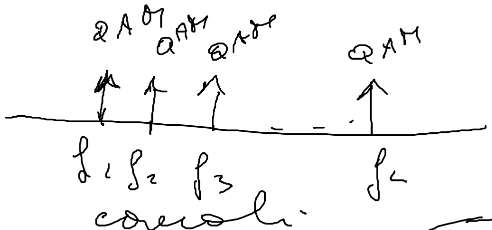


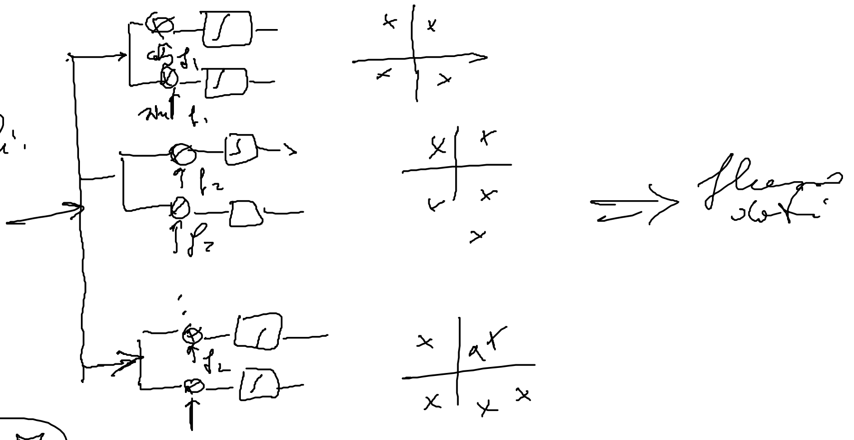
MT occupazione in banda ↑



QAM

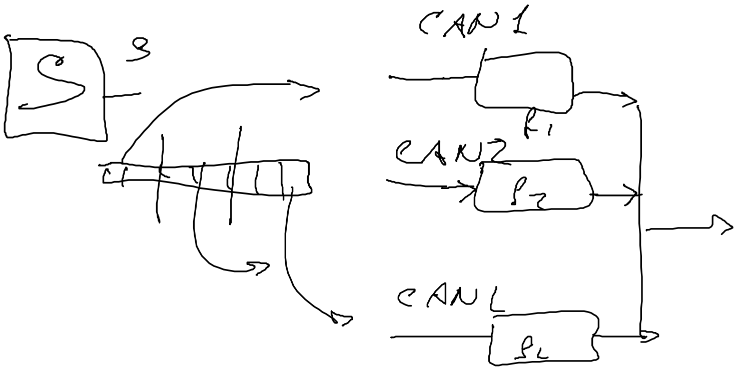
f_i → segnali ortogonali

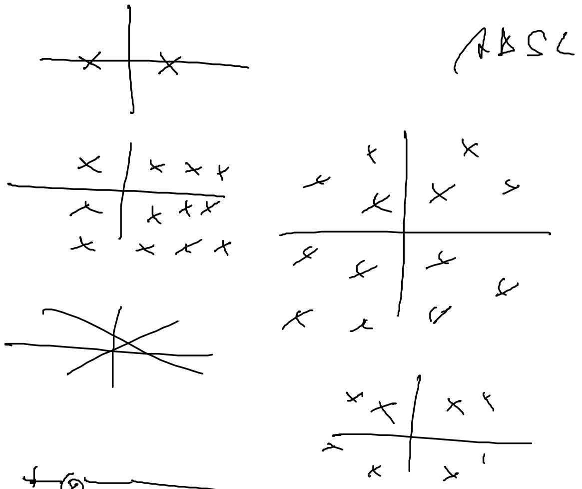


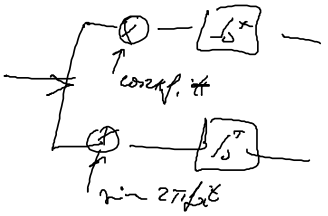


OFDM

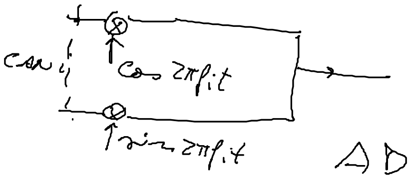
ORTHOGONAL FREQUENCY-DIVISION MOLTIPLEX





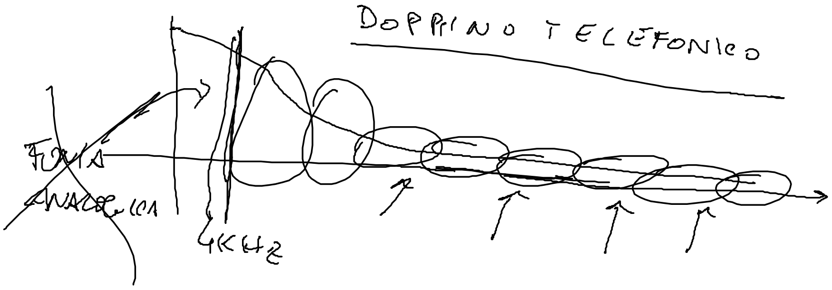


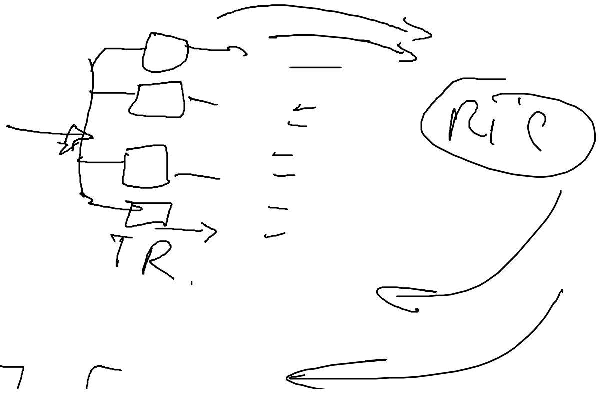
bit →

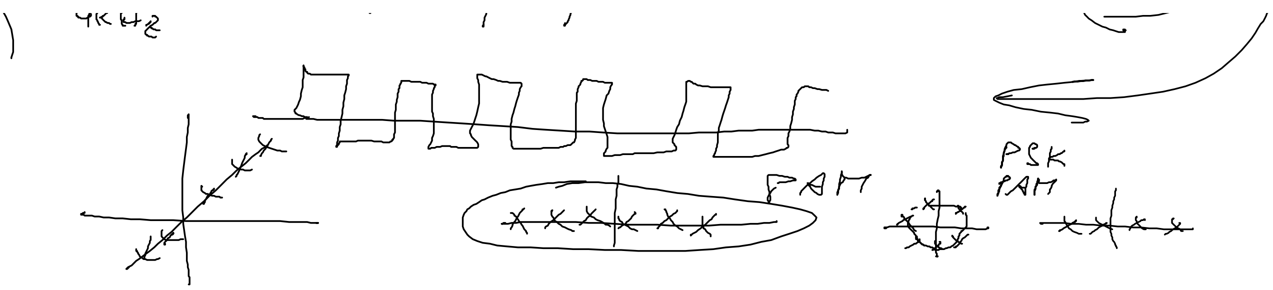


ADSL

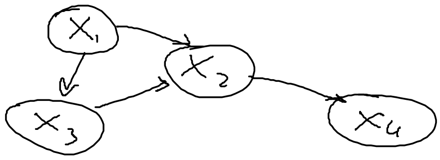
DOPPINO TELEFONICO

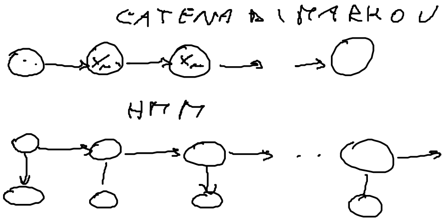


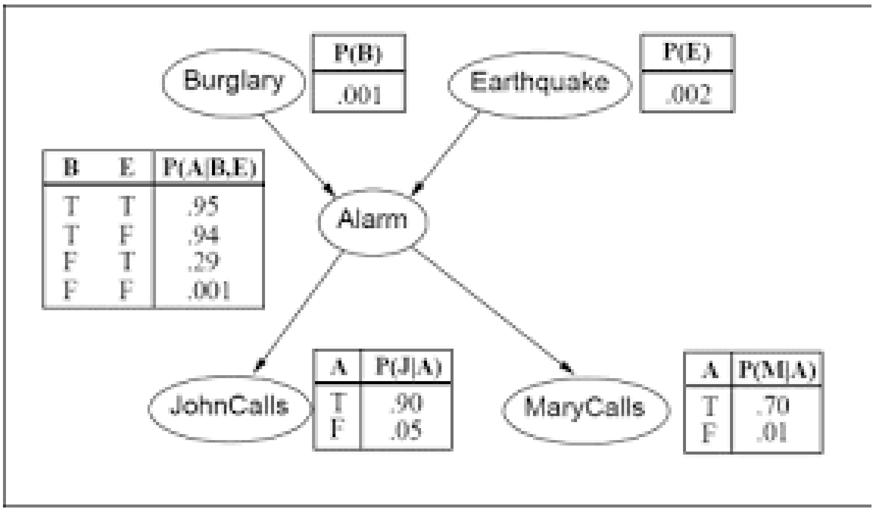




MODELLI BAYESIANI BASATI SU GRAFI







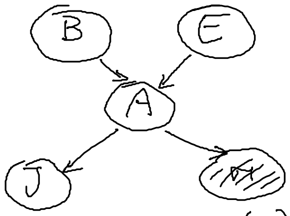
P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

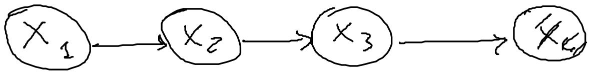
A	P(M|A)
T	.70
F	.01



$$p(A,B,E,J,M)$$
$$= p(A|BE)\,p(B)\,p(E)\,p(M|A)\,p(J|A)$$

$$p(E) = \sum_{A,B,J} p(A,B,E,J,\bar{M})$$

CATENA DI MARKOV



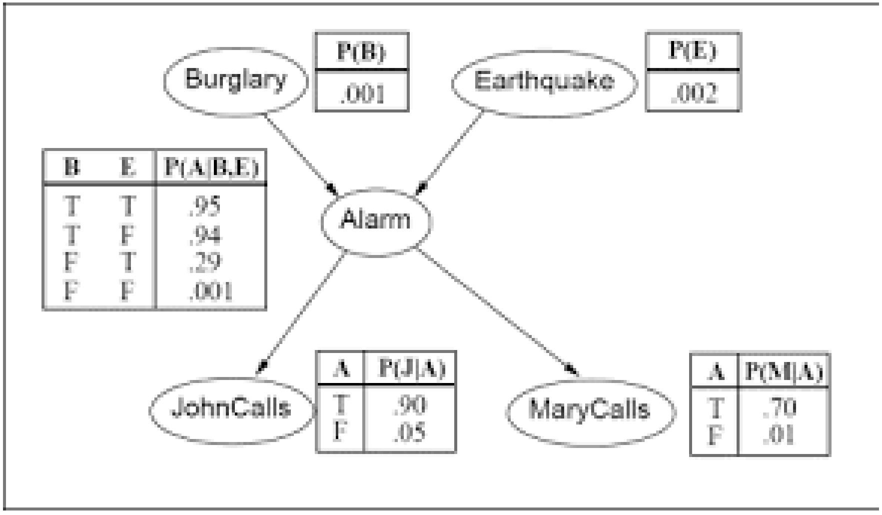
$$p(x_1 x_2 x_3 x_4) = p(x_4|x_3)\,p(x_3|x_2)\,p(x_2|x_1)\,p(x_1)$$

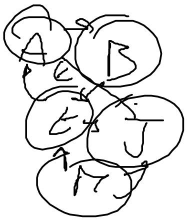
$$p(X_1|\bar{X}_4) = \sum_{X_2 X_3} p(x_1 X_2 X_3 \bar{X}_4)$$

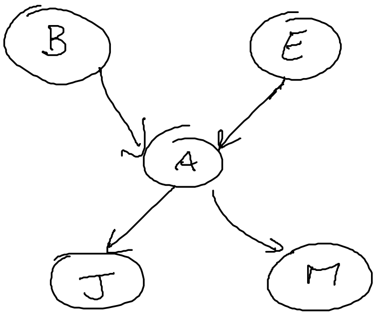
Lezione 23

lunedì 21 dicembre 2020 10.59

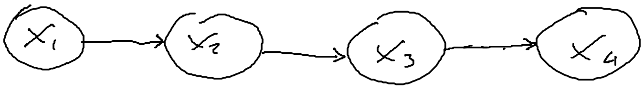
SISTEMI BAYESIANI



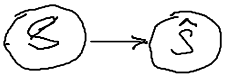




$$p(A,B,E,J,M) = p(B)p(E)\,p(A|BE)\,p(J|A)\,p(M|A)$$

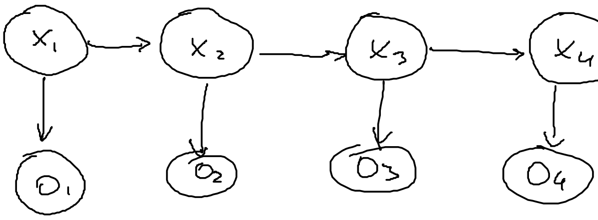


$$p(x_1 x_2 x_3 x_4) = p(x_4|x_3)\,p(x_3|x_2)\,p(x_2|x_1)\,p(x_1)$$



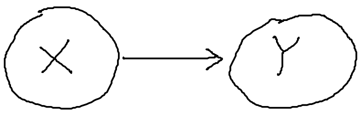
$$p(s,\hat{s}) = p(\hat{s}|s)\,p(s)$$

HMM



$$p(x_1\, x_2\, x_3\, x_4\, O_1\, O_2\, O_3 O_4) = p(x_1)\,p(O_1|x_1)\,p(x_2|x_1)\,p(O_2|x_2)\,p(x_3|x_2)\,p(O_3|x_3)\,p(x_4|x_3)\,p(O_4|x_4)$$

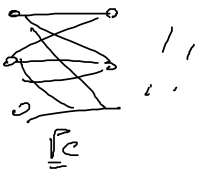
$$x \in \mathcal{X} = \{x^1, \ldots, x^n\}$$



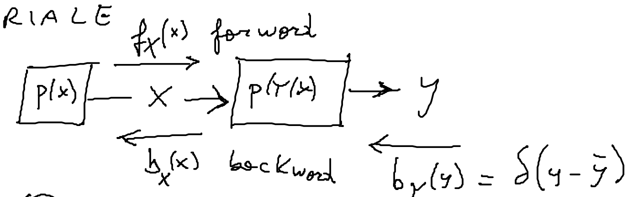
$X \in \mathcal{X} = \{x^1, \dots, x^N\}$

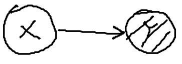
$Y \in \mathcal{Y} = \{y^1, \dots, y^M\}$

$$p(x,y) = p(Y|X)\, p(X)$$



GRAFO FATTORIALE





$Y = \bar{y}$

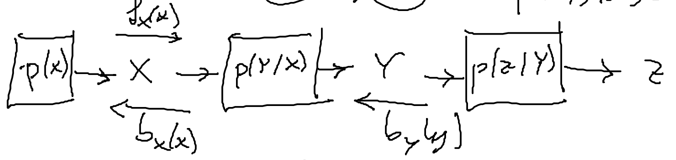
$$p(x|Y=\bar y) = \frac{p(x, Y=\bar y)}{p(Y=\bar y)} = \frac{p(Y=\bar y|x)\, p(x)}{p(Y=\bar y) \to \text{cost.}} \propto \underbrace{p(Y=\bar y|x)}_{b_X(x)} \underbrace{p(x)}_{f_X(x)}$$

$$b_X(x) = \int_{\mathcal{Y}} p(y|x)\, \delta(y-\bar y)\, dy = p(\bar y|x)$$ ↓ PRODUCT

SUM

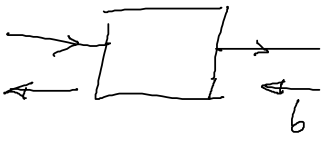
SUM - PRODUCT

$$p(x,y,z) = p(z|y)\,p(y|x)\,p(x)$$

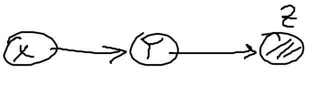


$$p(x,y,\bar z) = \int_{\mathcal{Y}} p(x,y,\bar z)\, dy$$

$$= \int_{\mathcal{Y}} p(\bar z|y)\, p(y|x)\, p(x)\, dy$$

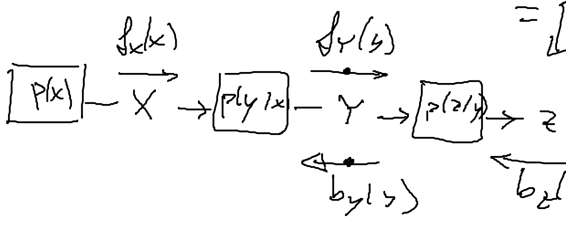


$$= \underbrace{p(x)}_{f_X(x)} \underbrace{\int \underbrace{p(\bar z|y)}_{b_Y(y)} p(y|x)\, dy}_{b_X(x)}$$

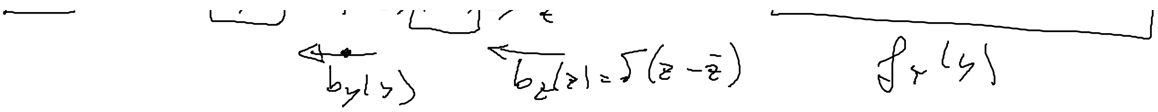


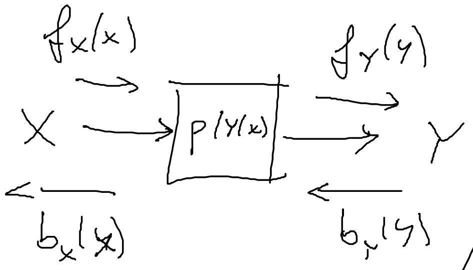
$$p(y|z=\bar z) = \frac{p(y, z=\bar z)}{p(z=\bar z)} \propto p(y, z=\bar z) = \int_{\mathcal{X}} p(x,y,z=\bar z)\, dx$$

$$= \int_{\mathcal{X}} p(z=\bar z|y)\, p(y|x)\, p(x)\, dx$$



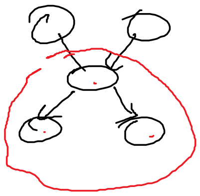
$$= \underbrace{p(z=\bar z|y)}_{b_Y(y)} \underbrace{\int_X p(y|x) \underbrace{p(x)}_{f_X(x)} dx}_{f_Y(y)}$$

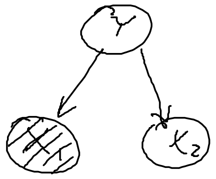




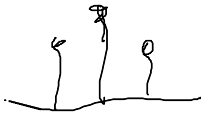
$$f_Y(y)=\int_{\mathcal{X}} p(y|x)\, f_X(x)\, dx$$

$$b_X(x)=\int_{\mathcal{Y}} p(y|x)\, b_Y(y)\, dy$$

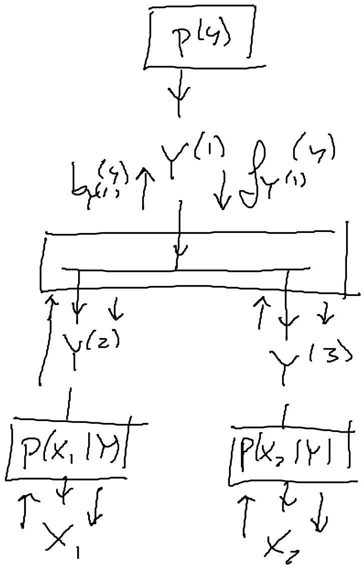




$$p(yX_1X_2)=p(X_1|Y)\,p(X_2|Y)\,p(y)$$



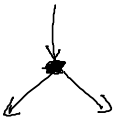
$b^{(y)}_{Y(2)}=p(\bar x_1|y)$



$b_{X_1}(x_1)=\delta(x_1-\bar x_1)$

$b_{X_2}(x_2)=U(x_2)$

REGOLA DEL PRODOTTO:

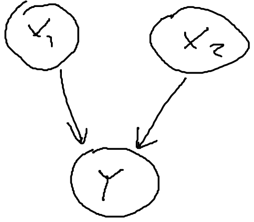


messaggio in uscita = prodotto dei messaggi in ingresso

$$b_{Y(3)}(y)=\int p(x_2|y)\,U(x_2)\,d(x_2 \propto U(y))$$

$$p(y|X_1=\bar x_1)=\frac{p(y,X_1=\bar x_1)}{p(X_1=\bar x_1)}\propto p(y,x_1=\bar x_1)=\int_{\mathcal{X}_2} p(y,X_1=\bar x_1,x_2)\,dx_2$$

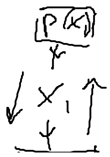
$$=\int_{\mathcal{X}_2} p(X_1=\bar x_1|y)\,p(x_2|y)\,p(y)\,dx_2=\underbrace{P(X_1=\bar x_1|y)}\ \underbrace{p(y)}_{f^{(y)}_{Y(1)}}\ \underbrace{\int_{\mathcal{X}_2} p(x_2|y)\,dx_2}_{1}$$

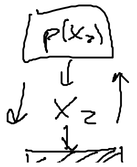


Due genitori

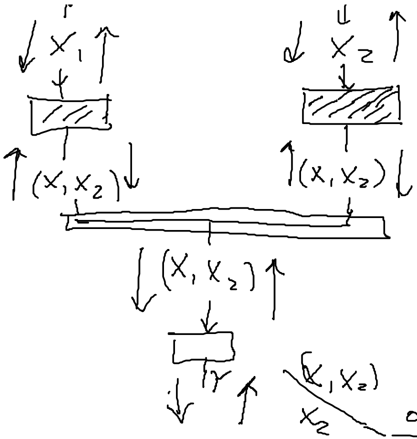
Figlio

$$p(Y,x_1,x_2)=p(Y|x_1,x_2)\,p(x_1)\,p(x_2)$$





$(X_1X_2)\in$ spazio prodotto $\mathcal{X}_1\times\mathcal{X}_2$



$\mathcal{X}_1 \times \mathcal{X}_2$

$p(x, x_2 | x_1) =$

$x_1 \in \mathcal{X}_1 = \{0, 1\}$

$x_2 \in \mathcal{X}_2 = \{0, 1\}$

(x_1, x_2) x_1

00 0

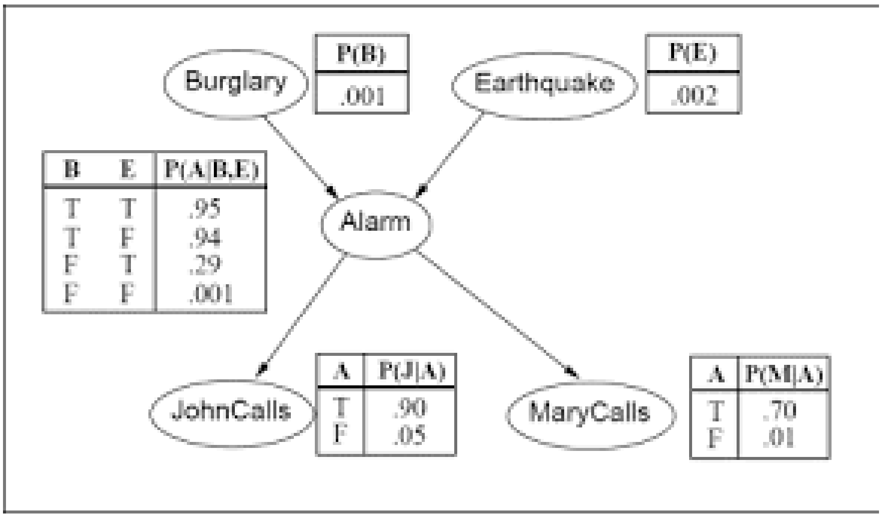
01 1

10

11

x_2 \ (x_1,x_2)	00	01	10	11
0	1/2	0	1/2	0
1	0	1/2	0	1/2

x_1 \ (x_1,x_2)	00	01	10	11
0	1/2	1/2	0	0
1	0	0	1/2	1/2



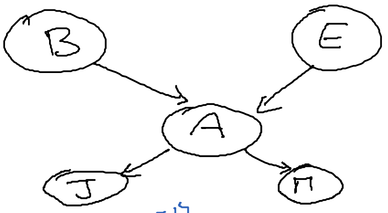
P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01



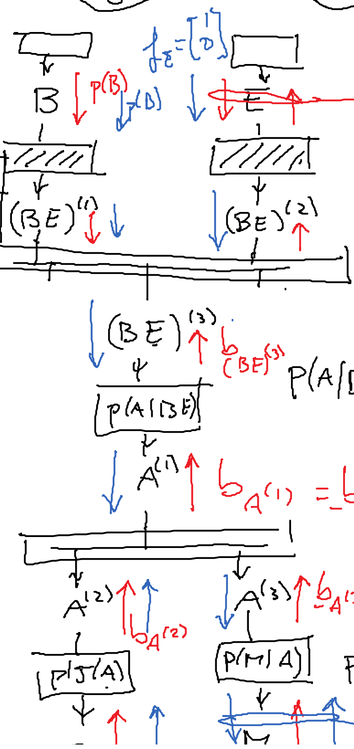
$p(B, E, A, J, M)$

$= p(A|BE)\, p(B)\, p(E)\, p(J|A)\, p(M|A)$

$p(B) = \begin{bmatrix} 0.01 \\ 0.99 \end{bmatrix}$

$p(E) = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix}$

B \ BE	TT	TF	FT	FF
T	1/2	1/2	0	0
F	0	0	1/2	1/2



E \ BE	TT	TF	FT	FF
T	1/2	0	1/2	0
F	0	1/2	0	1/2

$P(A|BE) =$

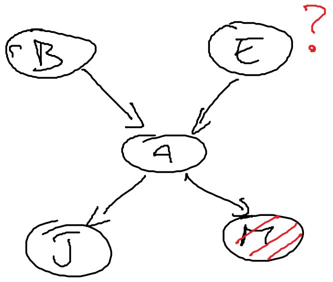
BE \ A	T	F
TT	0.95	0.05
TF	0.94	0.06
FT	0.29	0.61
FF	0.01	0.99

$P(J|A) =$

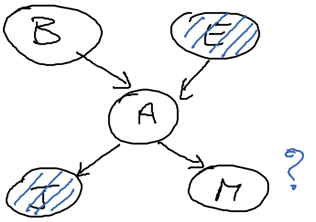
A \ J	T	F
T	0.9	0.1
F	0.01	0.99

$p(M|A) =$

A \ M	T	F
T	0.7	0.3
F	0.01	0.99



$p(e | M = \text{True})$



$P(J|A) =$

T	0.9	0.1
F	0.01	0.99

$\boxed{P(J|A)} \downarrow J$ $\quad \underline{b}_J = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ $\quad \underline{b}_J = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

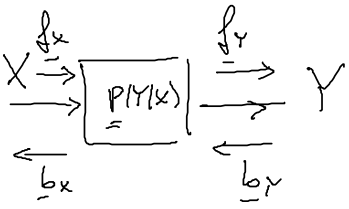
$\boxed{P(M|A)} \downarrow M$ $\quad b_M = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ $\quad \underline{b}_M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P(M|A) =$

T	0.7	0.3
F	0.01	0.99

Esposto

$$\underline{\underline{P}}_{Y|X} = \begin{array}{c|c} & Y \\ \hline X & \begin{pmatrix} \cdot 1 & \cdots & 1/2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \end{array}$$



$$\underline{f}_Y = \underline{\underline{P}}_{Y|X}^T \, \underline{f}_X$$

$$\underline{b}_X \propto \underline{\underline{P}}_{Y|X} \, \underline{b}_Y$$

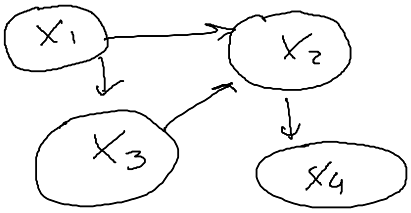
$$b_{A^{(3)}} \propto P_{M/A} \, \underline{b}_M = \begin{bmatrix} 0.7 & 0.3 \\ 0.01 & 0.99 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.01 \end{bmatrix}$$

$$b_{A^{(2)}} \propto \underline{\underline{P}}_{J/A} \, \underline{b}_J = \begin{bmatrix} 0.9 & 0.1 \\ 0.01 & 0.99 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

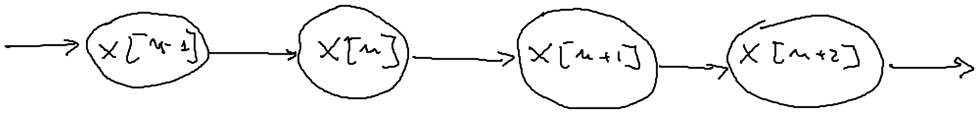
Lezione 24

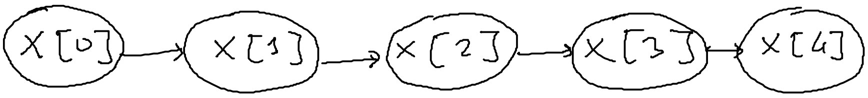
lunedì 4 gennaio 2021 11.04

RETI BAYESIANE



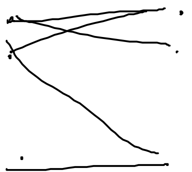
ESEMPIO HMM Hidden Markov Model



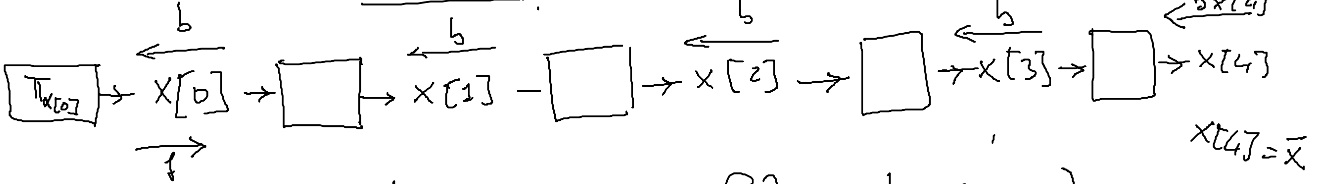


$X[n] \in \{a_1, a_2 .. a_m\}$

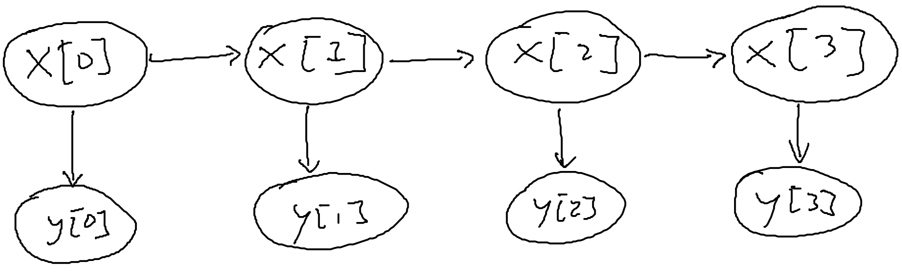
$p(X[n] \mid X(n-1))$



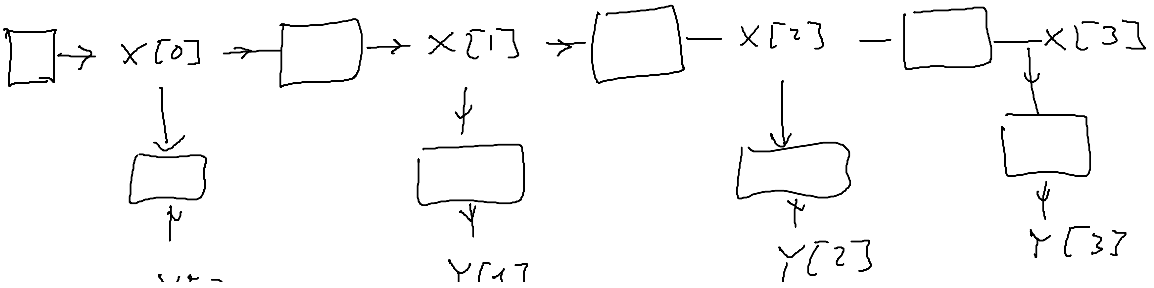
GRAFO FATTORIALE

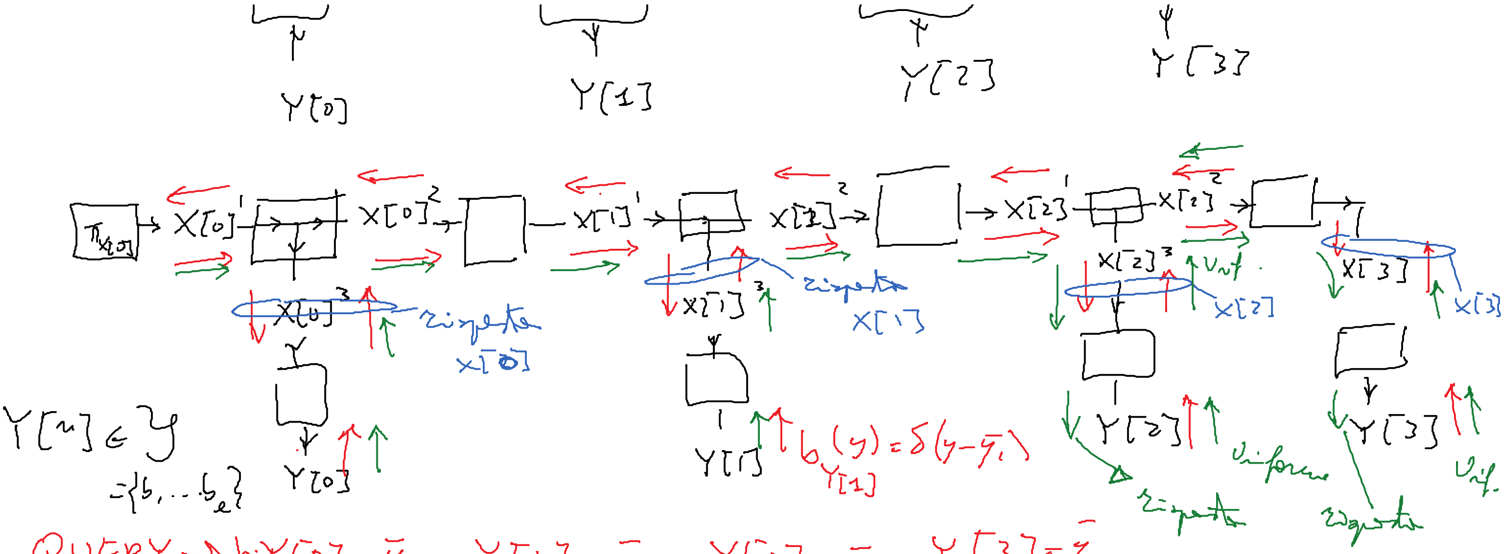


QUERY: Dato $X[4] = \bar{x}$, $P_z\{X[0] \mid X[4] = \bar{x}\}$



$$\begin{cases} X[k+1] = f(X[k]) \\ Y[k] = g(X[k]) \end{cases}$$





QUERY = Dato: $Y[0] = \bar{y}_0$, $Y[1] = \bar{y}_1$, $Y[2] = \bar{y}_2$, $Y[3] = \bar{y}_3$

calcolare la sequenza degli stati.

(prob.a posteriori)

$$\overset{?}{\Longrightarrow} p(x[0], x[1]\ x[2]\ x[3] \,|\, Y[0] = \bar{y}_0, Y[1] = \bar{y}_1, Y[2] = \bar{y}_2, Y[3] = \bar{y}_3)$$

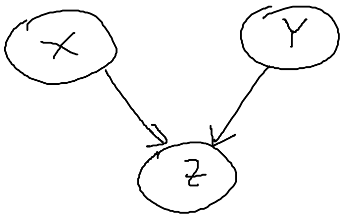
QUERY = Dato $Y[0] = \bar{y}_0$, $Y[1] = \bar{y}_1$ $\Longrightarrow p[Y[2] \,|\, Y[0] = \bar{y}_0, Y[1] = \bar{y}_1)$

Esempio

$X \in \{0, 1\}$ π_x $Y \in \{0, 1\}$ π_y

$Z = X + Y$

$Z \in \{0, 1, 2\}$

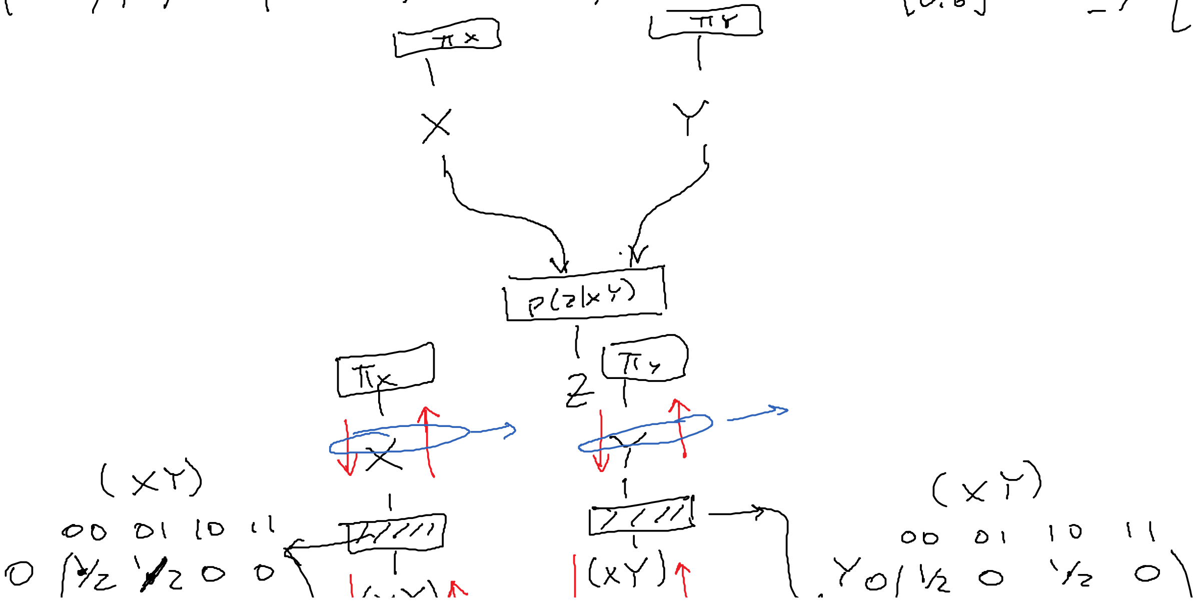


$P(z/xY) =$

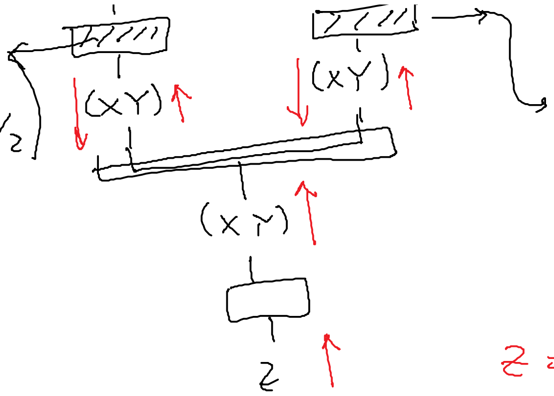
XY \ Z	0	1	2
0 0	1	0	0
0 1	0	1	0
1 0	0	1	0
1 1	0	0	1

$$\pi_x = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \qquad \pi_y = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$p(x, y, z) = p(z \,|\, x\, y)\, \pi(x)\, \pi(y)$$

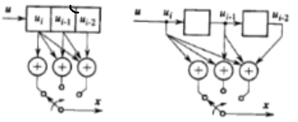


$$X\;\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$



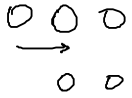
$$Y\;\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

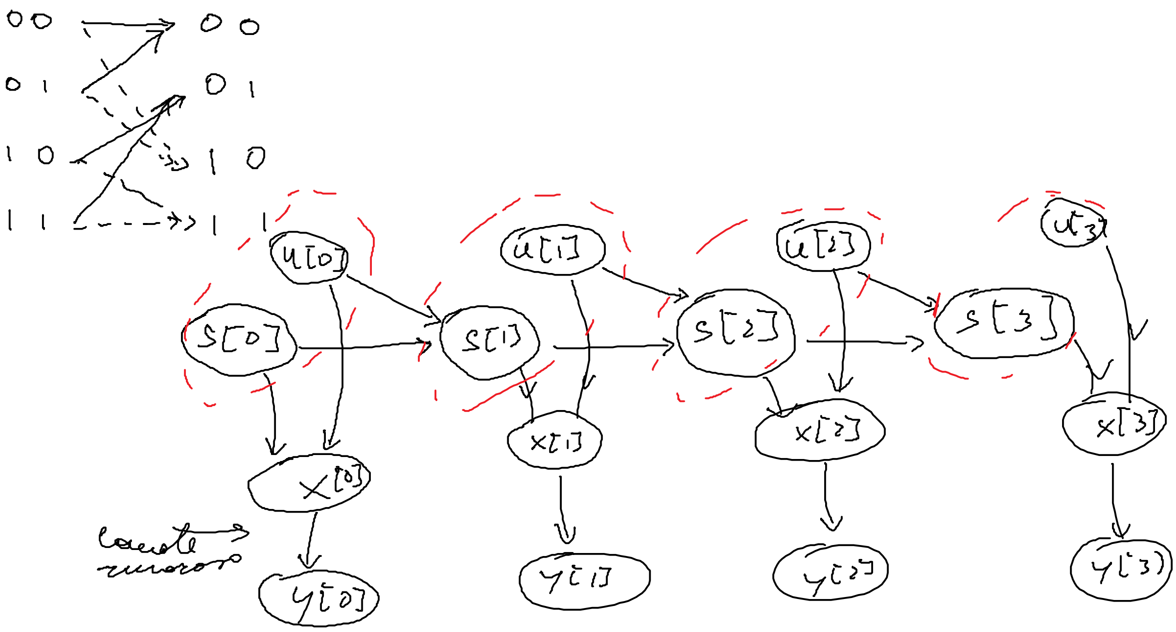
$z = \bar{z}$



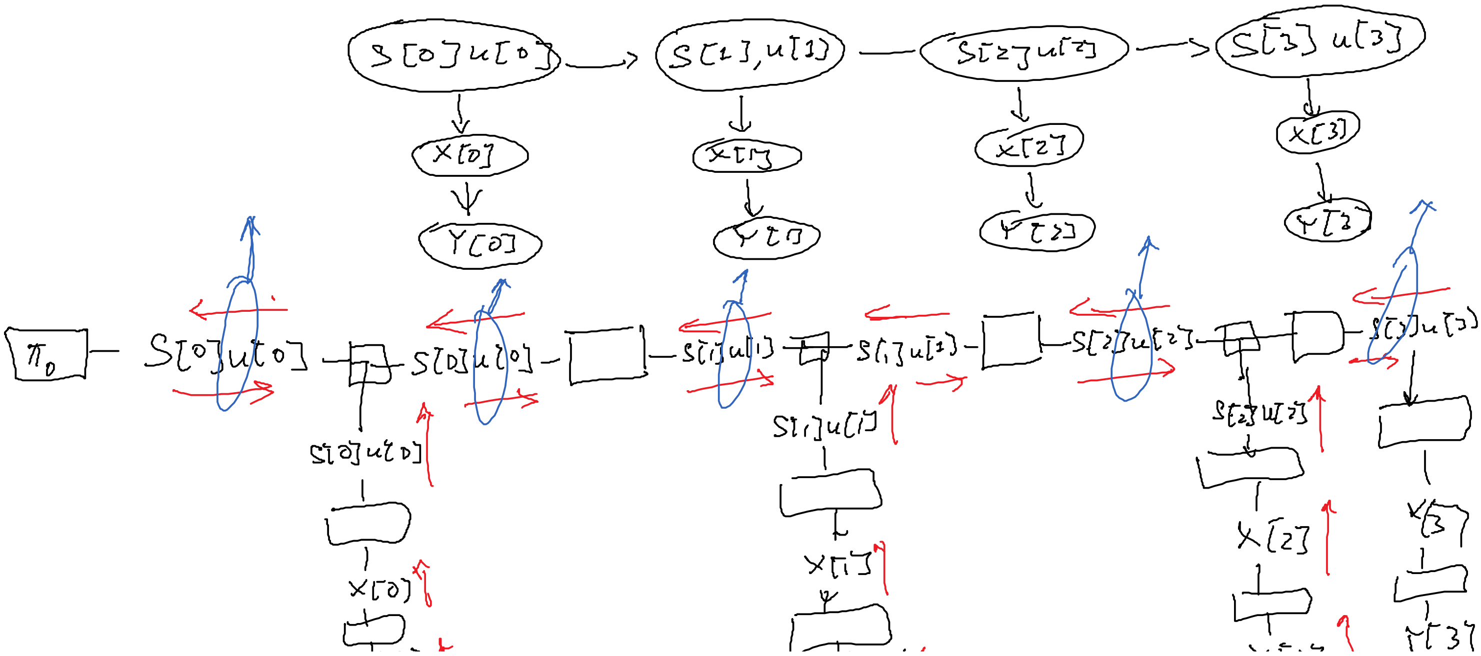
$u \in \{0,1\}$

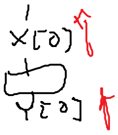
$x \in \{$stringa di 3 bit$\}$

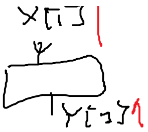


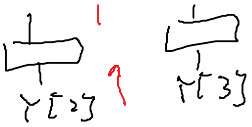


QUERY: Dato $y[0] = \bar{y}_0$, $Y[1] = \bar{y}_1$, $Y[2] = \bar{y}_2$, $Y[3] = \bar{y}_3$









Cenni SU Codifica LDPC
Cenni su Codifica Turbo