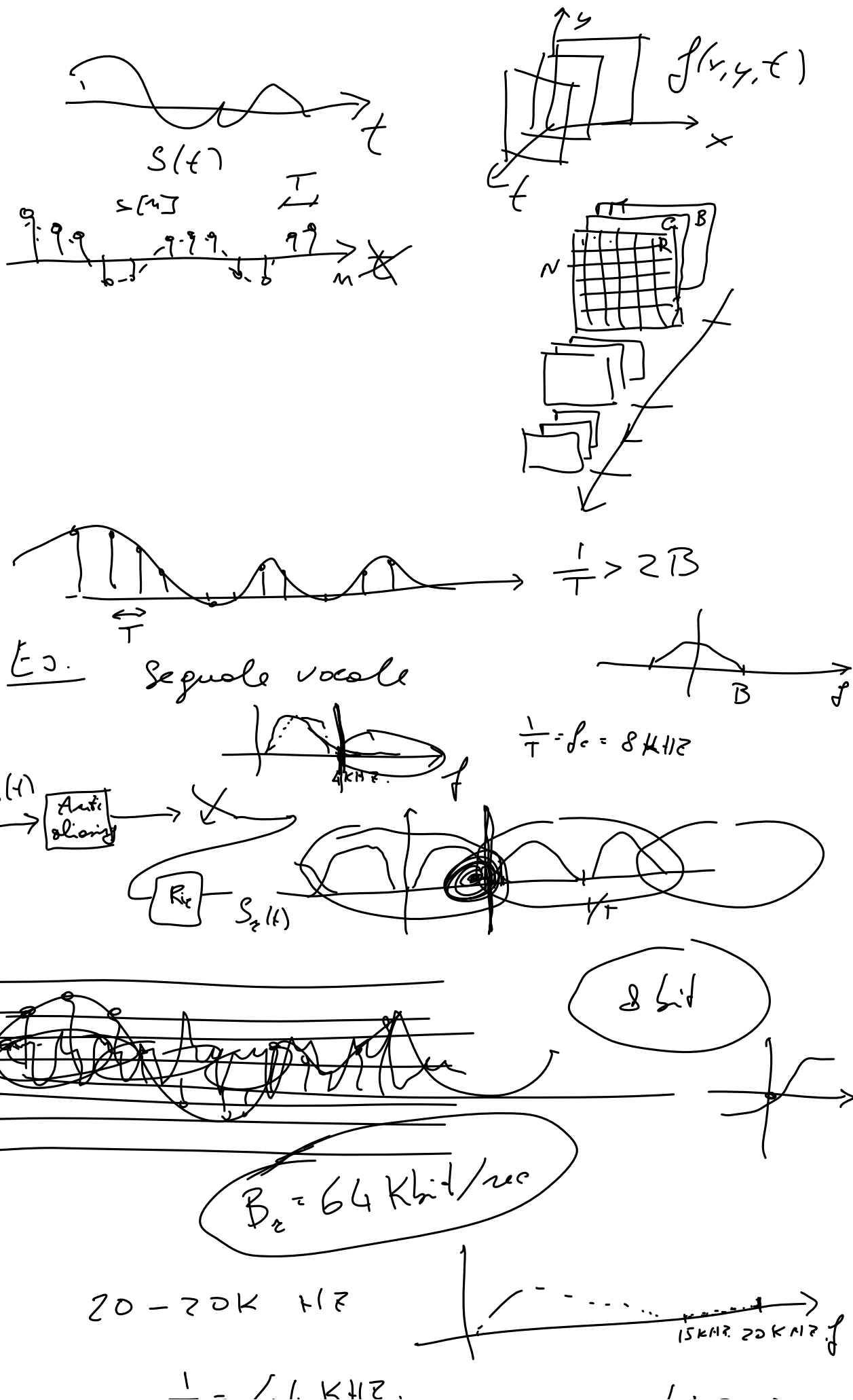


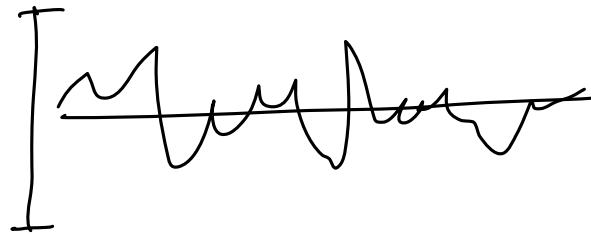


$$\frac{1}{T} = 44 \text{ KHz.}$$

16 bit

24 bit

48 bit



$$\frac{44000}{16}$$

$$\frac{264}{44}$$

$$\frac{704.000 \times}{2}$$

$$\frac{704.000 \times}{2}$$

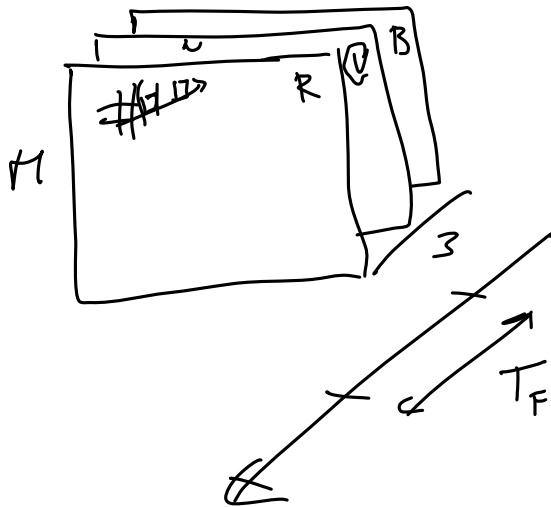
$$\frac{704.000 \times}{2}$$

$$\frac{704.000 \times}{2}$$

$$\frac{704.000 \times}{2}$$

compression.

MP3



$$N \cdot T \cdot 3 \cdot 8 = 1000 \cdot 500 \cdot (3 \cdot 8)$$

$$12000$$

$$= 12'000'000 \text{ Mbit.}$$

$$\frac{24 \times}{5}$$

$$\frac{120}{120}$$

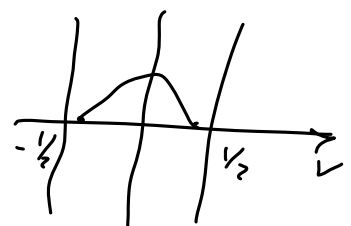
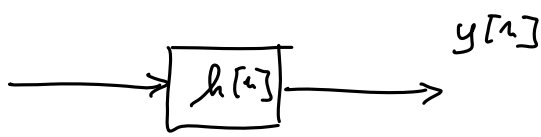
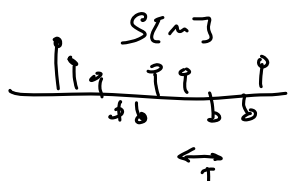
$$2^{24} =$$

$$B_2 = \frac{N \cdot T \cdot 3 \cdot 8}{T_F} =$$

$$T_F = \frac{1}{30}$$

$$= 12'000'000 \times 30$$

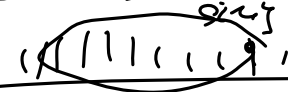
$$= 360'000'000$$



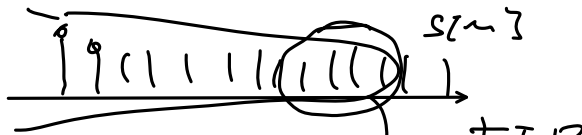
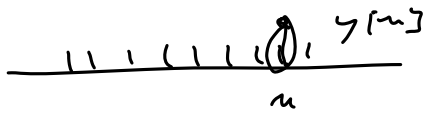
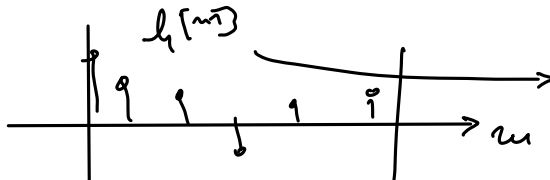
$$y[n] = (h \times s)[n] = \sum_{m=-\infty}^{+\infty} h[m] s[n-m]$$

casuale

$$h[m] = 0 \quad m < 0$$

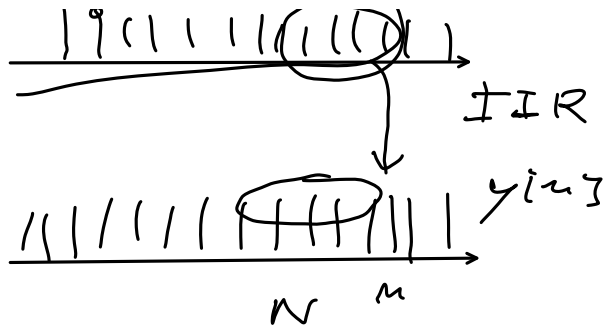


IIR
FIR



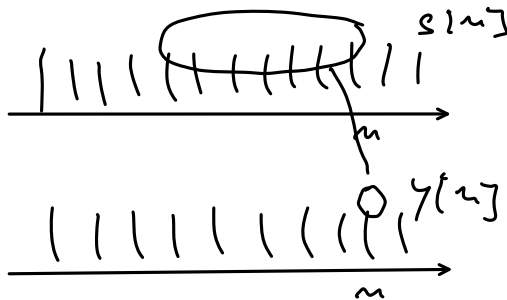
$$y[n] = \sum_{m=0}^{\infty} h[m] s[n-m]$$

IIR



$$y[n] = \sum_{m=1}^M a_m y[n-m] + \sum_{k=0}^{N-1} b_k x[n-k]$$

FIR



$$y[n] = \sum_{m=0}^{N-1} h[m] s[n-m]$$

FIR

$$y[n] = \sum_{m=0}^{N-1} b_m s[n-m]$$

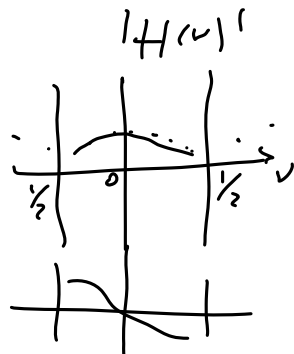
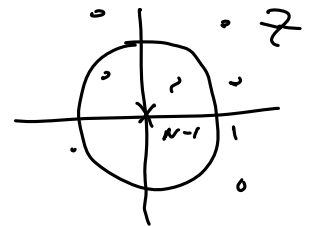
$$Y(z) = \sum_{m=0}^{N-1} b_m z^{-m} S(z)$$

$$H(z) = \frac{Y(z)}{S(z)} = \sum_{m=0}^{N-1} b_m z^{-m}$$

$$= z^{-N+1} (b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_{N-1})$$

$$= \frac{1}{z^{N-1}} (b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_{N-1})$$

$$H(\nu) = \sum_{m=0}^{N-1} b_m e^{-j2\pi\nu m} = \sum_{m=0}^{N-1} b_m (\cos 2\pi\nu m - j \sin 2\pi\nu m)$$



IIR

$$y[n] = \sum_{m=1}^M a_m y[n-m] + \sum_{k=0}^{N-1} b_k x[n-k]$$

$$Y(z) = \sum_{m=1}^M a_m z^{-m} Y(z) + \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

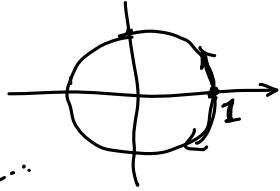
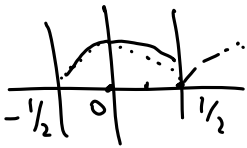
$$Y(z) (1 - \sum_{m=1}^M a_m z^{-m}) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{m=1}^M a_m z^{-m}} = \frac{z^{-N+1} (b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_{N-1})}{z^{-M} (z^M - a_1 z^{M-1} - a_2 z^{M-2} - \dots - a_M)}$$

$$= z^{-N+M+1} \frac{\left(\begin{matrix} \cdot \\ \cdot \end{matrix} \right)}{\left(\begin{matrix} \cdot \\ \cdot \end{matrix} \right)}$$

$$H(\nu) = H(z) \Big|_{e^{j2\pi\nu}} = \frac{\sum_{k=0}^{N-1} b_k e^{-j2\pi\nu k}}{1 - \sum_{m=1}^M a_m e^{-j2\pi\nu m}}$$

freq. ()
→ 17 pol.

(E3)

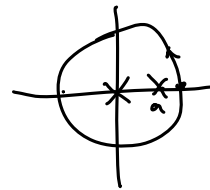
$$y[n] = a y[n-1] + b x[n-2]$$

$$Y(z) = a z^{-1} Y(z) + b z^{-2} X(z)$$

$$H(z) = \frac{b z^{-2}}{1 - a z^{-1}} = \frac{z^{-2} b}{z^{-1} (z - a)} = \frac{1}{z} \frac{b}{z - a}$$

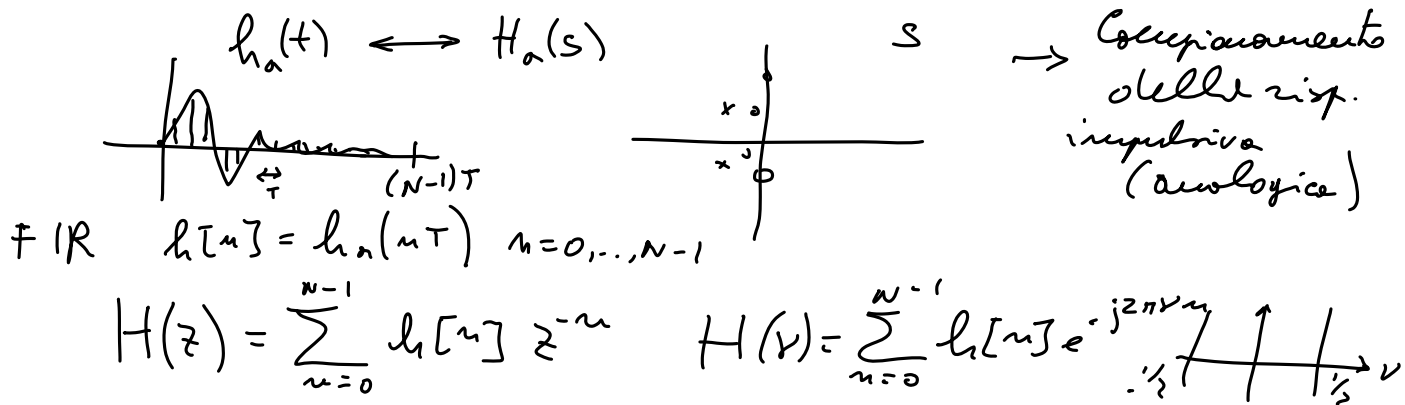
$z = a$

$$H(\nu) = \frac{b e^{-j2\pi\nu 2}}{1 - a e^{-j2\pi\nu}}$$



$FIR \rightarrow$ Serie di Fourier
 $IIR \rightarrow$ Progettazione Poli-Zeri

$\{a_k, b_k\}$



$h_a(t) = e^{-\alpha t} u(t)$

$H_a(s) \leftrightarrow h_a(t)$

$h[n] = h_a[nT] \quad n=0, \dots, \infty$

$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \mathcal{Z} \left[\mathcal{L}^{-1} [H_a(s)] \right]_{t=nT}$

$H_a(s) = \frac{1}{s + \alpha}$

$H(z) = \sum_{n=0}^{\infty} e^{-\alpha nT} z^{-n} = \sum_{n=0}^{\infty} (e^{-\alpha T} z^{-1})^n = \frac{1}{1 - e^{-\alpha T} z^{-1}} \quad IIR$

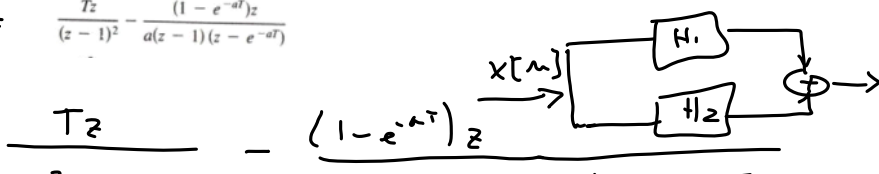
$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1$

$y[n] = e^{-\alpha T} y[n-1] + x[n]$

$h_a(t) \leftrightarrow H_a(s)$

$H(z) = \mathcal{Z} \left[\mathcal{L}^{-1} [H_a(s)] \right]_{t=nT}$

$H(z) = \frac{Tz}{(z-1)^2} - \frac{(1-e^{-\alpha T})z}{\alpha(z-1)(z-e^{-\alpha T})}$



$$\frac{Tz}{z^2 - 2z + 1} = \frac{(1 - e^{-aT})z}{a(z^2 - z(1 + e^{-aT}) + e^{-aT})}$$

$$H_1(z) = \frac{Tz^{-1}}{1 - 2z^{-1} + z^{-2}}$$

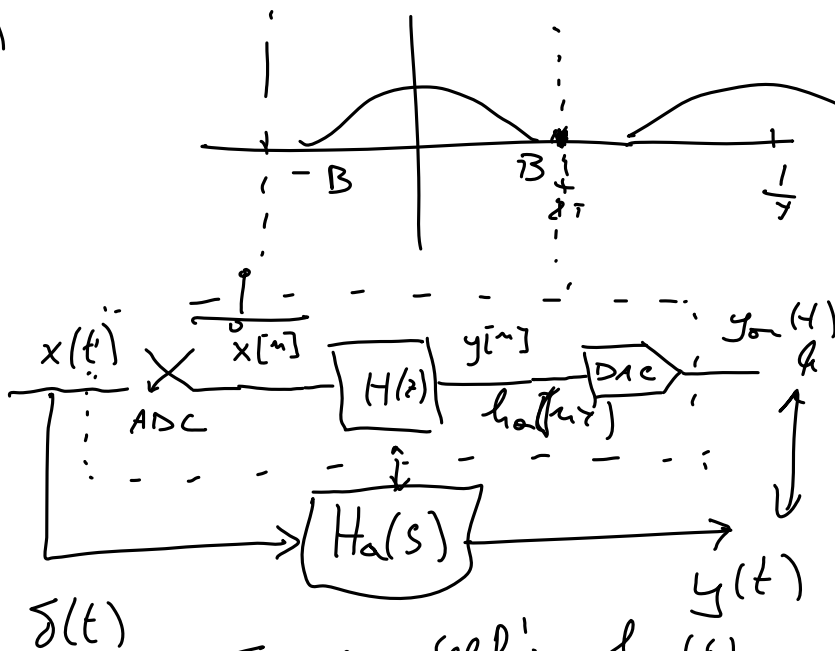
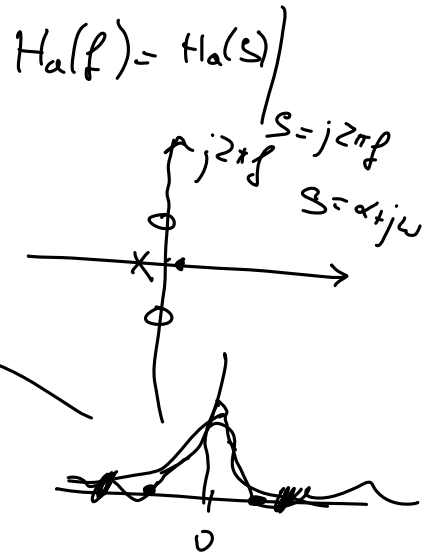
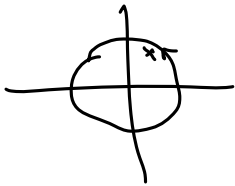
$$H_2(z) = \frac{\frac{1}{a}(1 - e^{-aT})z^{-1}}{1 - (1 + e^{-aT})z^{-1} + e^{-aT}z^{-2}}$$

$$y[n] = (1 + e^{-aT})y[n-1] - e^{-aT}y[n-2] + \frac{1}{a}(1 - e^{-aT})x[n]$$

$$y[n] = 2y[n-1] - y[n-2] + Tx[n]$$

$$\frac{Y(z)}{X(z)} = \frac{Tz^{-1}}{1 - 2z^{-1} + z^{-2}}$$

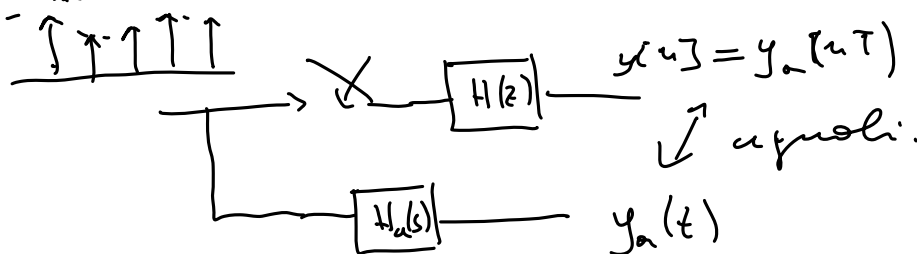
$$y[n] - 2y[n-1] + y[n-2] = Tx[n]$$



Tecnica dell'invarianza all'impulso

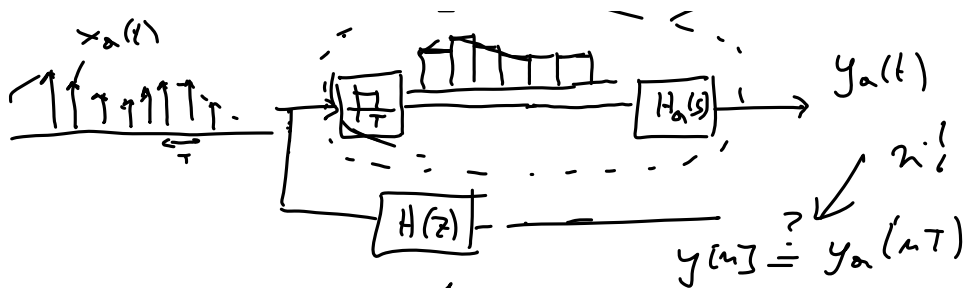
non le stesse
 e l'ingresso è un impulso ideale $\delta(t)$

$$\sum_{n=-\infty}^{+\infty} x_a(nT) \delta(t - nT)$$



$$h_{sh}(t) = \Pi\left(\frac{t - T/2}{T}\right)$$

$$= u(t) - u(t - T)$$

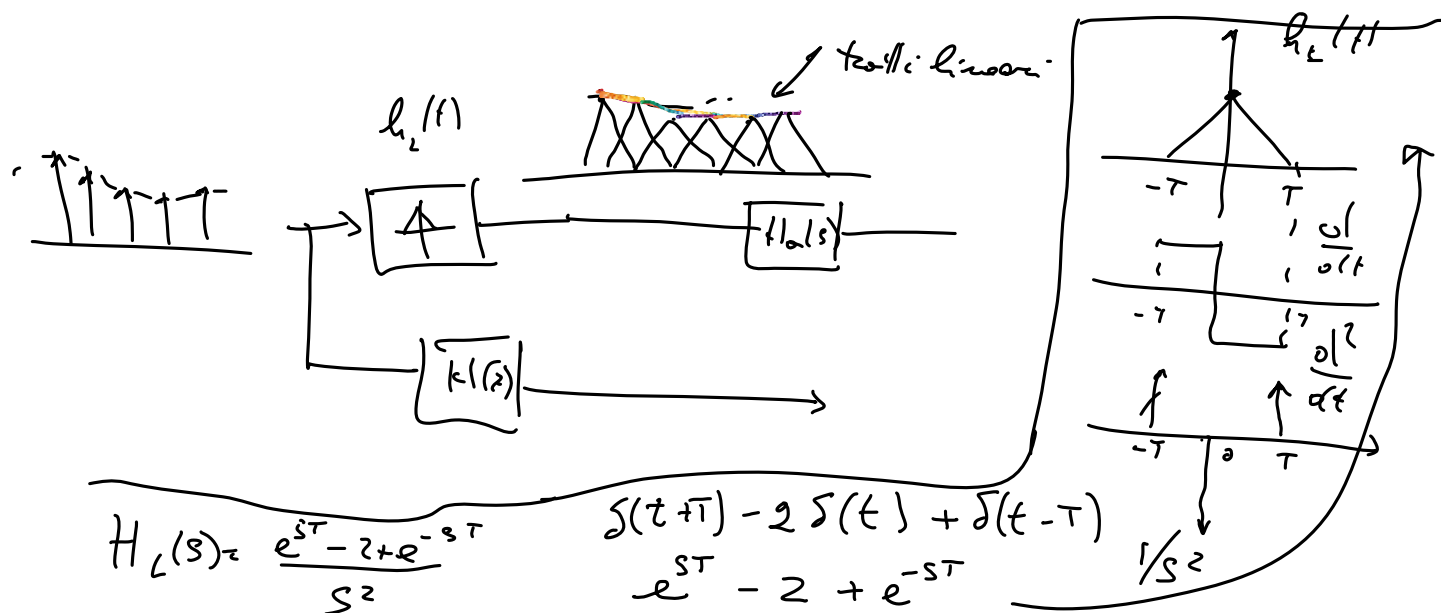


$$H_{SH}(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1 - e^{-sT}}{s}$$

$$H(z) = \mathcal{Z} \left[\mathcal{L}^{-1} \left[H_a(s) \left(\frac{1 - e^{-sT}}{s} \right) \right] \right]_{t=nT}$$

$$= (1 - z^{-1}) \mathcal{Z} \left[\mathcal{L}^{-1} \left[\frac{H_a(s)}{s} \right] \right]_{t=nT}$$

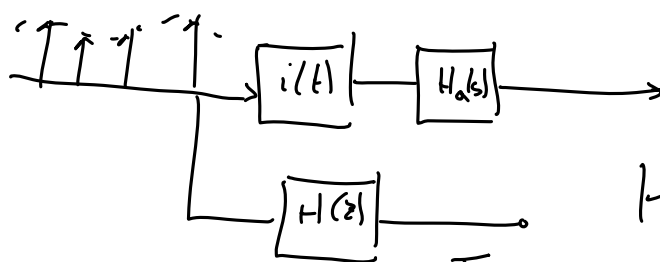
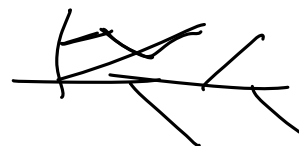
TECNICA DI
INVARIANZA
AL GRADINO



$$H(z) = \mathcal{Z} \left[\mathcal{L}^{-1} \left[H_a(s) \frac{e^{sT} - 2 + e^{-sT}}{s^2} \right] \right]_{t=nT}$$

$$= (z - 2 + z^{-1}) \mathcal{Z} \left[\mathcal{L}^{-1} \left[\frac{H_a(s)}{s^2} \right] \right]_{t=nT}$$

Tecnica dell'inv.
alla rampa



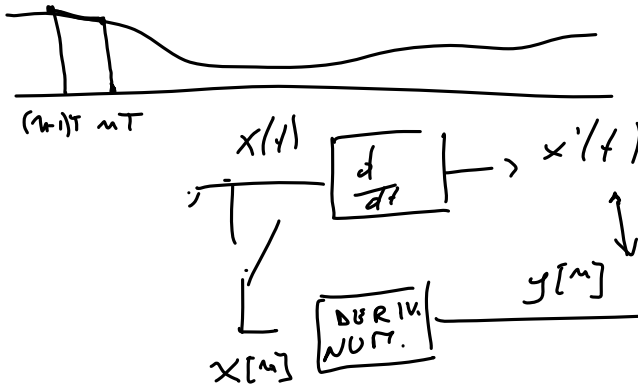
$$H(z) = \mathcal{Z} \left[\mathcal{L}^{-1} \left[H_a(s) I(s) \right] \right]_{t=nT}$$

Tecnica dell'inv. all'impulso
o ad una classe di interp.



$$\sum_k \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k_i} \beta_{k_i} \frac{d^{k_i} x(t)}{dt^{k_i}}$$

$$H_a(s) = \frac{\sum \beta_k s^{k_i}}{\sum \alpha_k s^k}$$



$$y[n] = \frac{x[n] - x[n-1]}{T}$$

$$Y(z) = X(z) \left(\frac{1 - z^{-1}}{T} \right)$$

$$H_a(s) = \frac{A}{s + \alpha}$$

$$H(z) = \frac{A}{\frac{1 - z^{-1}}{T} + \alpha} = \frac{A}{1 - z^{-1} + \alpha T} = \frac{AT}{(1 + \alpha T) - z^{-1}} = \frac{AT}{(1 + \alpha T)} \frac{1}{1 - \frac{1}{1 + \alpha T} z^{-1}}$$

$$y[n] = \frac{1}{1 + \alpha T} y[n-1] + \frac{AT}{1 + \alpha T} x[n]$$

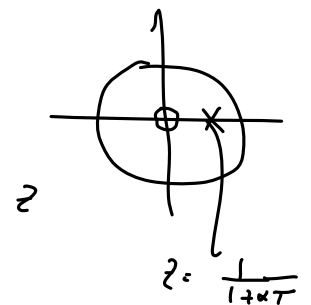
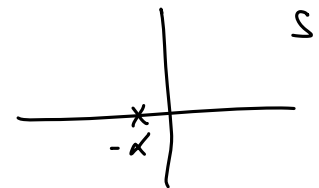
$$S \leftarrow \frac{1 - z^{-1}}{T}$$

$$S = \frac{1 - z^{-1}}{T} \rightarrow ST = \frac{z - 1}{z}$$

$$H_a(s) \xrightarrow{zST = z - 1} \boxed{\frac{1}{z - 1}} \rightarrow H(z)$$

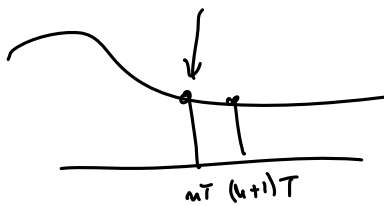
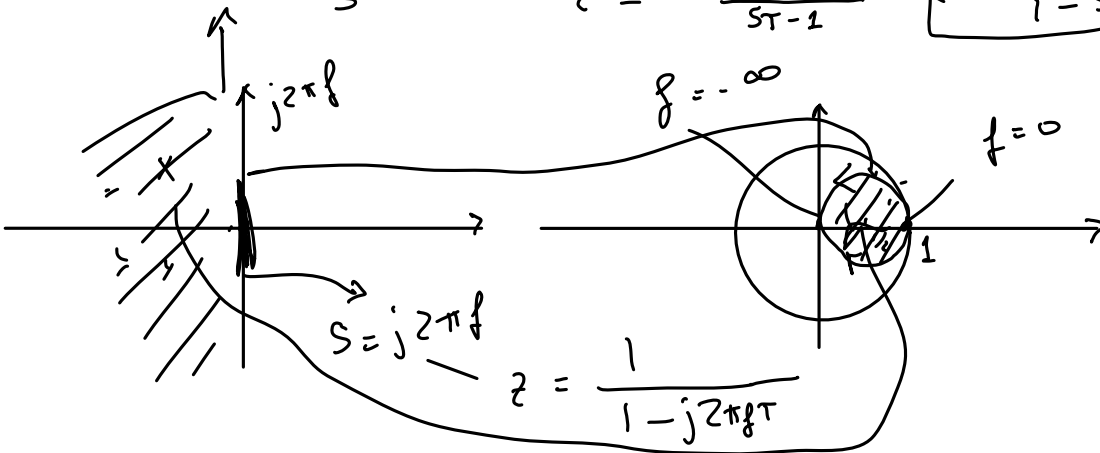
$$S \left(H(z) = H_a(s) \right) \left|_{S = \frac{1 - z^{-1}}{T}} \right.$$

$$D(z) = \frac{Y(z)}{X(z)} = \left(\frac{1 - z^{-1}}{T} \right)$$



$$H_A(s) \xrightarrow{s} z \text{ ST} = z - 1 \quad \boxed{z = \frac{1}{1 - sT}} \rightarrow h(z)$$

$$z = -\frac{1}{sT - 1}$$



$$\frac{x[n+1] - x[n]}{T}$$

ANTIRIAUSALĖ

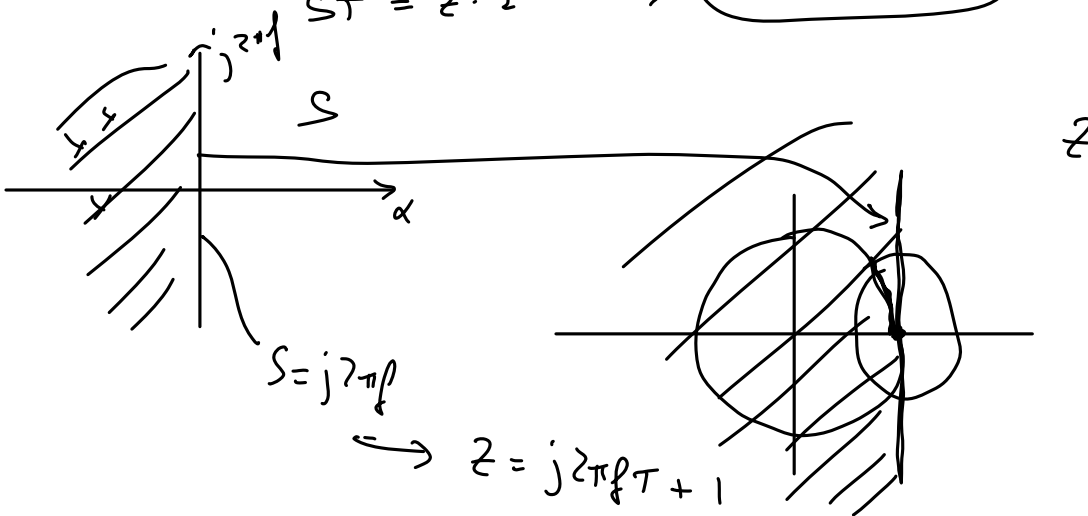
$$H_A(s) \Big|_{s = \frac{z-1}{T}} = H(z)$$

$$D(z) = \frac{z - 1}{T}$$

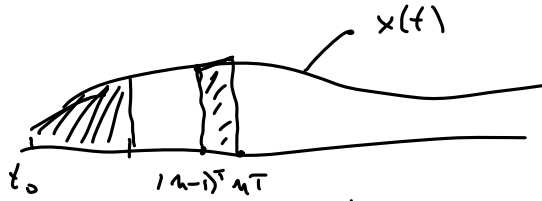
$$\boxed{s \leftarrow \frac{z - 1}{T}}$$

c

$$sT = z - 1 \rightarrow \boxed{z = sT + 1}$$

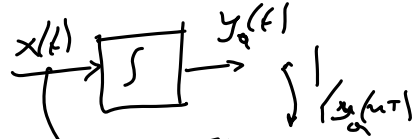


$$s = \sigma + j2\pi f \rightarrow z = \sigma + j2\pi fT + 1$$

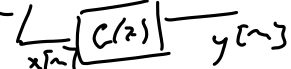


$$\frac{1}{s} \rightarrow G(z)?$$

$$y(t) = y(t_0) + \int_{t_0}^t x(\xi) d\xi$$



$$y(nT) = y((n-1)T) + \left(\frac{x(nT) + x((n-1)T)}{2} \right) T$$



$$y[n] = y[n-1] + \frac{T}{2} (x[n] + x[n-1])$$

$$Y(z) = z^{-1} Y(z) + \frac{T}{2} X(z) + \frac{T}{2} z^{-1} X(z)$$

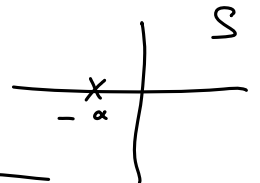
$$G(z) = \frac{Y(z)}{X(z)} = \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}} \rightarrow \frac{1}{s}$$

$$s \leftarrow \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = H_a(s) \Big|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

$$H_a(s) = \frac{1}{s + \alpha}$$

$$0 < \alpha < 1$$



$$H(z) = \frac{1}{\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} + \alpha} = \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1} + \frac{\alpha T}{2} (1+z^{-1})}$$

$$= \frac{T}{2} \frac{1+z^{-1}}{(1 + \frac{\alpha T}{2}) + (\frac{\alpha T}{2} - 1) z^{-1}}$$

$$= \frac{\frac{T}{2}}{2(1 + \frac{\alpha T}{2})} \frac{1+z^{-1}}{1 - \frac{(\frac{\alpha T}{2} - 1)}{(\frac{\alpha T}{2} + 1)} z^{-1}}$$

$$y[n] = -a y[n-1] + g x[n] + g x[n-1]$$

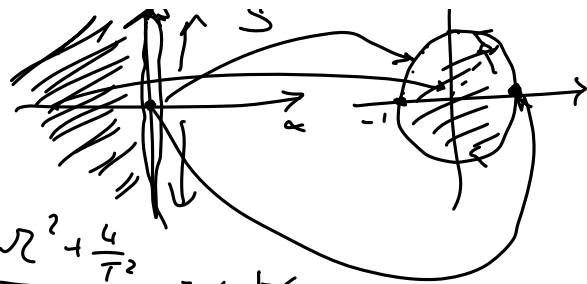
$$\Omega = 2\pi f \text{ rad/sec.}$$

$$z = re^{j\omega}$$

$$z = -\frac{s + \frac{\alpha}{T}}{s - \alpha}$$



$$z = -\frac{s + \frac{2}{T}}{s - \frac{2}{T}}$$

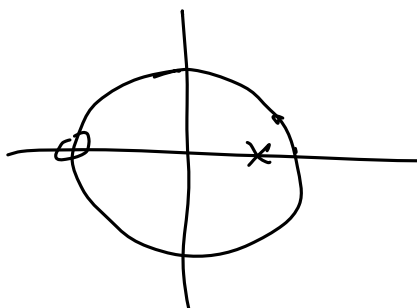
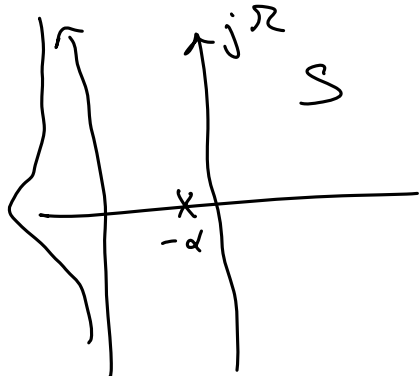
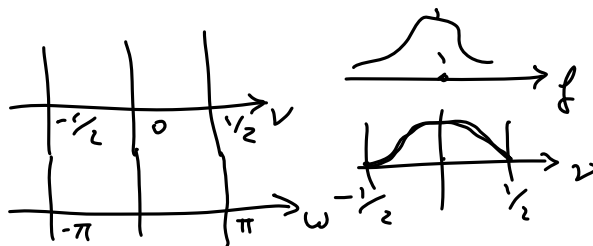


$$s = j\Omega \rightarrow -\frac{j\Omega + \frac{2}{T}}{j\Omega - \frac{2}{T}}$$

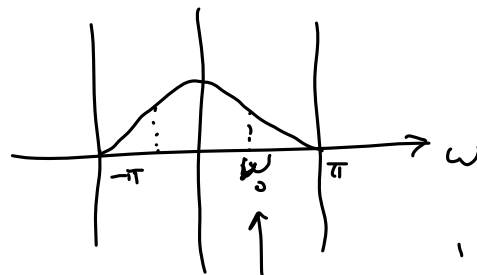
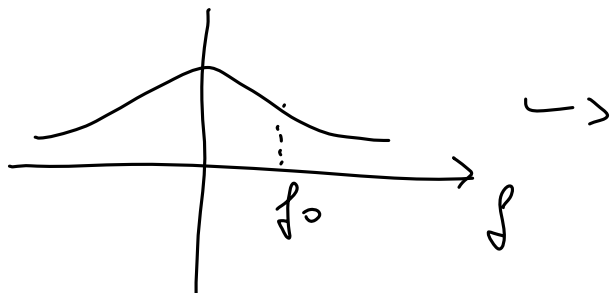
$$|z|^2 = \frac{\Omega^2 + \frac{4}{T^2}}{\Omega^2 + \frac{4}{T^2}} = 1 \quad \forall \Omega$$

$$e^{j\omega} = -\frac{j\Omega + \frac{2}{T}}{j\Omega - \frac{2}{T}}$$

$$\omega = \angle \frac{-j\Omega + \frac{2}{T}}{j\Omega - \frac{2}{T}} =$$



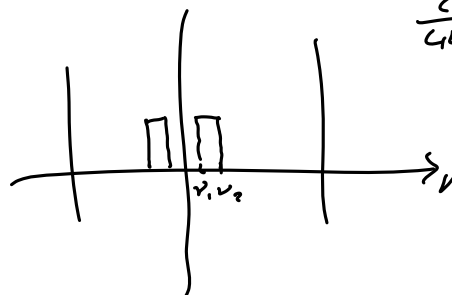
$$\frac{1}{2} = \frac{f}{f_c}$$



$$\omega_0 = 2\sqrt{\frac{2}{T}} \frac{\Omega_0 T}{2}$$

$$\Omega_0 = \frac{2}{T} \sqrt{\frac{2}{T}} \frac{\omega_0}{2}$$

$$\frac{1}{2} = \frac{f}{f_c} \quad \frac{1}{5} \text{ KHz.} \quad \frac{2}{44000} \quad \frac{1}{44000}$$

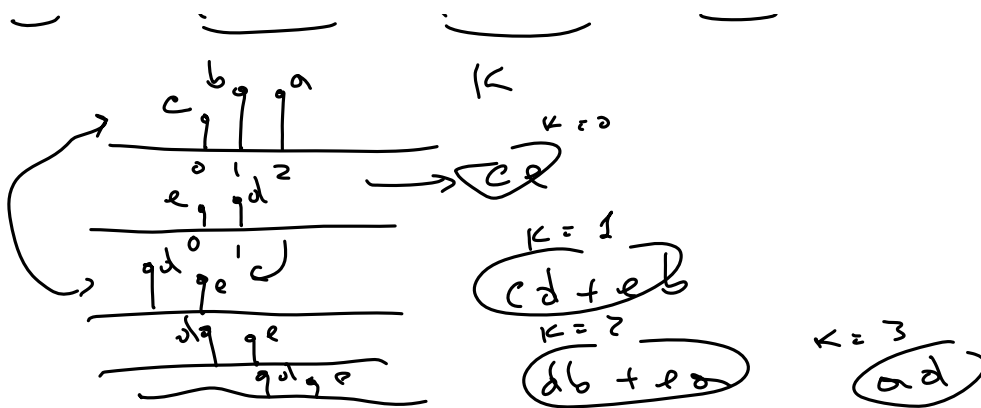


$$(ax^2 + bx + c)(dx + e)$$

$$ad x^3 + \underbrace{bd} x^2 + \underbrace{cd} x + \underbrace{ae} x^2 + \underbrace{be} x + \underbrace{ce}$$

$$\underbrace{ad} x^3 + \underbrace{(bd + ae)} x^2 + \underbrace{(cd + be)} x + \underbrace{ce}$$

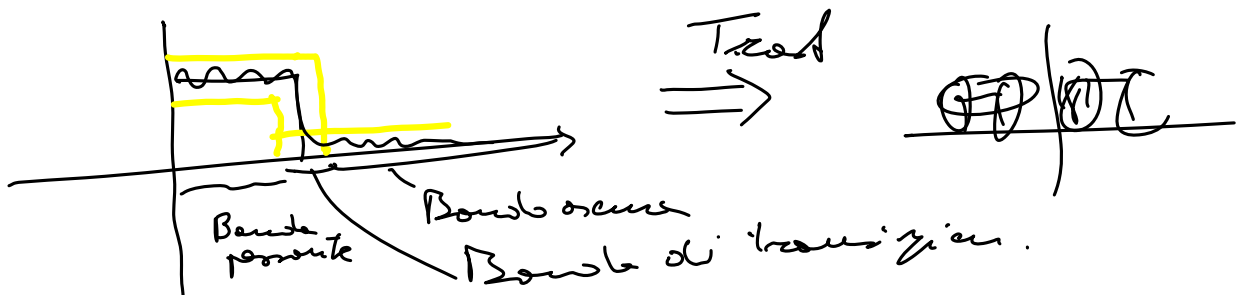
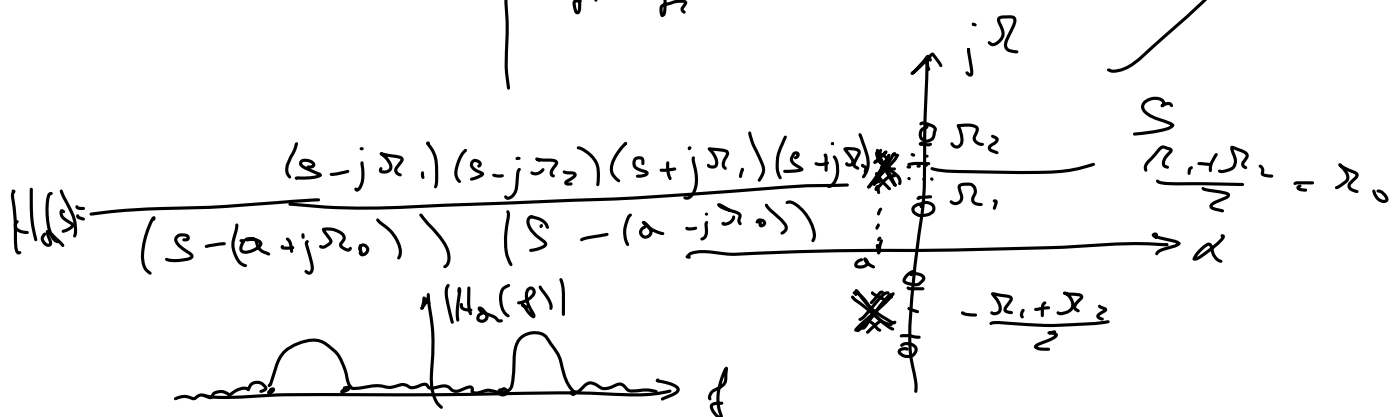
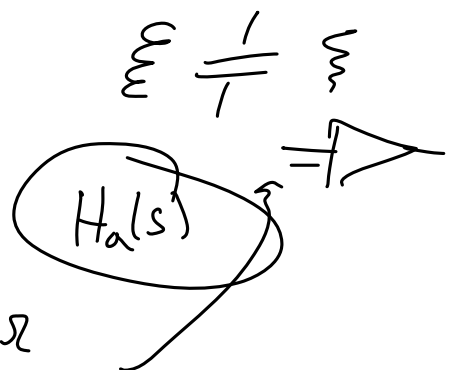
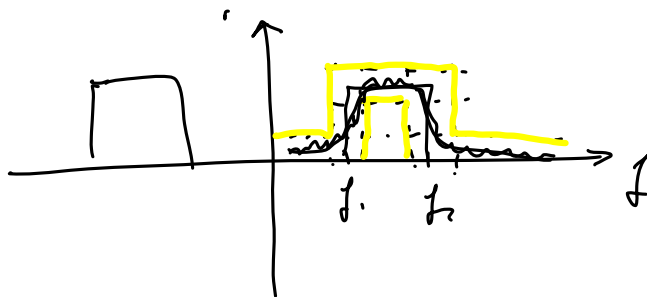
$$c \quad b \quad a \quad K$$



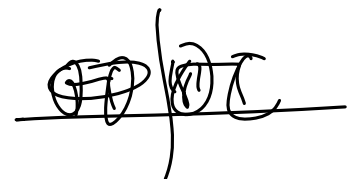
$$(3x^2 + 2x + 1)(-x + 1)$$

$$-3x^3 + 2x^2 - x + 3x^2 + 2x + 1$$

$$-3x^3 + x^2 + x + 1$$



$\Rightarrow$ Tread



PASSA-BASSO ANALOGICO

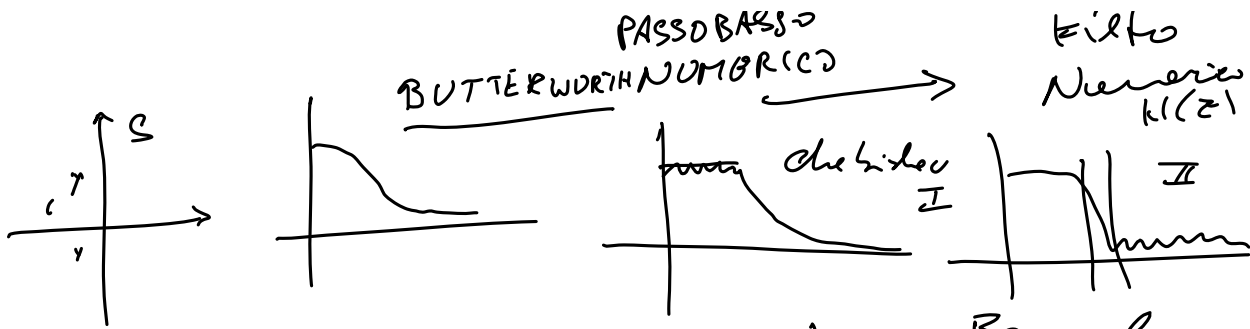
PASSO BASSO

BUTTERWORTH NUMERICO

PASSA BANDA ANALOGICO

FILTO

NUMERICO



$$\left| H_{LP}(z) \right|^2 = \frac{1}{1 + \left(\frac{z}{z_c} \right)^{2N}}$$

$$\begin{cases} \rho = z_c \\ \theta = \frac{\pi(N+1)}{2N} + k \frac{2\pi}{2N} \end{cases}$$

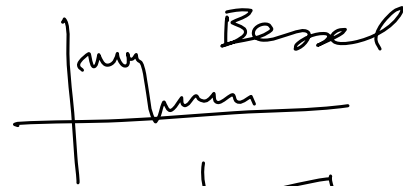
$$\begin{cases} \rho = z_c \\ \theta_k = \frac{\pi}{2} + \frac{\pi}{2N} + \frac{\pi}{N} k \\ k = 0, \dots, 2N-1 \end{cases}$$

POLI DI

$$H_{LP}(s) \cdot H_{LP}(-s)$$

poli di $H_{LP}(s)$ stabili.

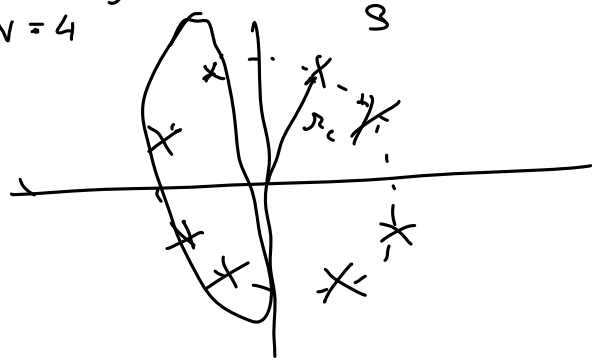
$$H_{LP}(s) = \frac{(-1)^4 s_0 s_1 s_2 s_3}{(s - s_0)(s - s_1)(s - s_2)(s - s_3)}$$



$$j\Omega = s$$

$$\Omega = \frac{s}{j}$$

$$N=4$$



PASSA BASSO

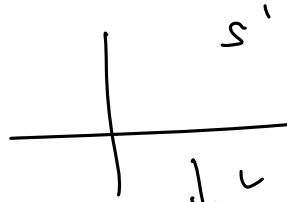
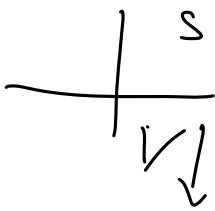
$$H_{LP}(s)$$



PASSA BANDA

Analitico

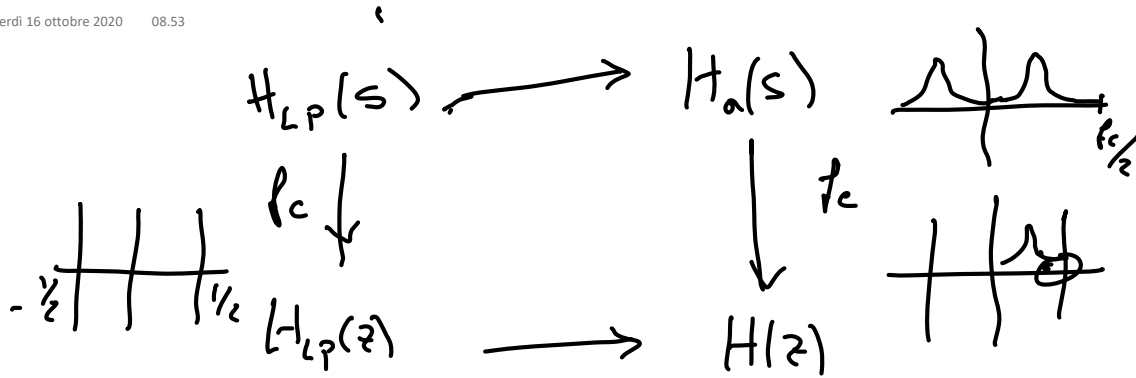
$$H_a(s)$$



$H_{LP}(z)$ PASSA BASSO NUMERIC

$$z \leftarrow f(z')$$

$$H(z')$$



TRASFORMAZIONI DA
FILTRO PASSA-BASSO
ANALOGICHE E NUMERICHE

Lezione 4: Trasmissioni
Prof. FRANCESCO A.N. PALLER

CORSO DI TRASMISSIONI ED ELABORAZIONE
NUMERICA DEI SEGNALE - SUN
AA. 2014-15

TRASFORMAZIONI (filtri analogici)

Filtro Analogico
Passa Basso
 $H_{LP}(s)$

Filtro $H_A(s)$
Analogico
- passa basso
- passa alto
- passa banda
- a banda estrema.

Otteniamo un filtro passa-basso dallo
specifiche di progetto, per passare facilmente
ad un filtro di natura diversa esiste
delle semplici formule di trasformazione.

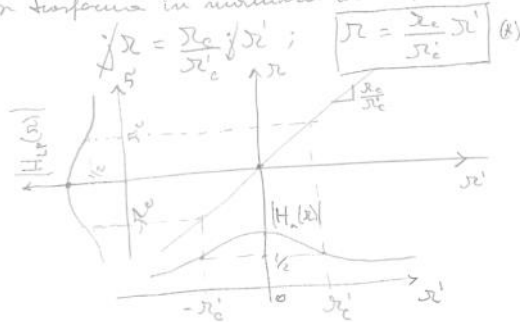
In particolare il filtro analogico finale viene
ottenuto per semplice sostituzione.

$$H_A(s') = H_{LP}(s) \Big|_{s=g(s')}$$

Esaminiamo le 4 principali formule di trasformazione

PASSA-BASSO freq. di taglio ω_c	$\Rightarrow$	PASSA-BASSO freq. di taglio ω'_c
$s = \frac{\omega_c}{\omega'_c} s'$		

La trasformazione è la più semplice e realizza una
semplice riscalaatura del piano s . In particolare l'asse
 $j\omega$ si trasforma in un retta invisibile nell'asse $j\omega'$



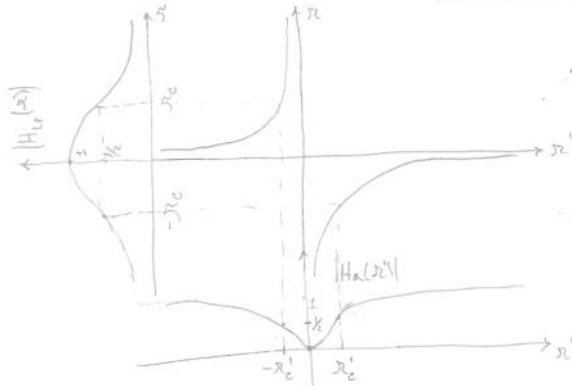
È evidente come in questo caso sia semplice verificare che se $H_L(s)$ è stabile (poli a parte reale negativa) anche $H_A(s)$ è stabile. Il piano sinistro di s si mappa direttamente nel piano sinistro di s' .
 Se il filtro passa-basso è stato progettato con una maschera con specifiche di attenuazione a ω_p e ω_s , le nuove ω'_p e ω'_s possono essere ottenute direttamente dalla formula (*) : $\omega' = \frac{\omega_c}{\omega} \omega$

PASSA-BASSO freq. di taglio ω_c	$\Rightarrow$	PASSA ALTO freq. di taglio ω'_c
$s = \frac{\omega_c \omega'_c}{s'}$		

La trasformazione è ora non lineare. Abbiamo un vantaggio che il nuovo ω' è direttamente proporzionale a ω . Infatti per $s' = j\omega'$

$$s = \frac{\omega_c \omega'_c}{j\omega'} = -j \frac{\omega_c \omega'_c}{\omega'} = j\omega$$

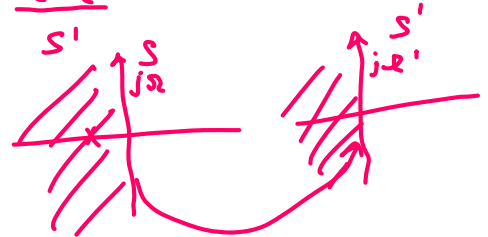
La trasformazione delle frequenze è $\omega = -\frac{\omega_c \omega'_c}{\omega'} \quad (\omega \neq 0)$



Ora la banda passante si ritrova a destra di ω'_c e la banda oscura al di sotto.

$$H_{LP}(s) \rightarrow H(s')$$

$$s = \frac{\omega_c \omega'_c}{s'}$$

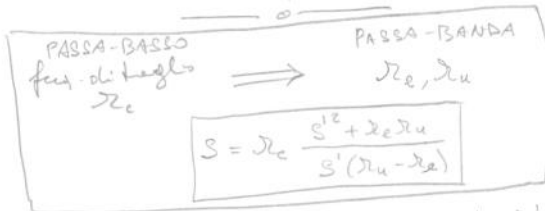
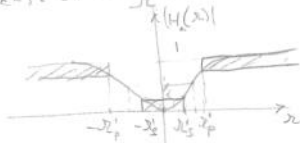


Per verificare la convergenza della stabilità basta scrivere $_{TAS}$

$$S' = \frac{Z_c Z_c'}{S} = \frac{Z_c Z_c'}{s + j\omega} = \frac{Z_c Z_c' (s - j\omega)}{s^2 + \omega^2}$$

$$\alpha' + j\omega' = \frac{Z_c Z_c' \alpha'}{\alpha'^2 + \omega'^2} - \frac{Z_c Z_c' \omega'}{\alpha'^2 + \omega'^2} \quad \left(\begin{array}{l} \alpha' \text{ ed } \omega' \text{ hanno lo} \\ \text{stesso segno} \\ \text{quindi il prodotto } \alpha' \omega' \\ \text{è sempre negativo} \end{array} \right)$$

Le specifiche del filtro passa-banda in termini di Z_p e Z_s vanno tradotte in termini di Z_e e Z_u come in figura da formula (44): $Z_e' = -\frac{Z_e Z_s}{Z_p}$

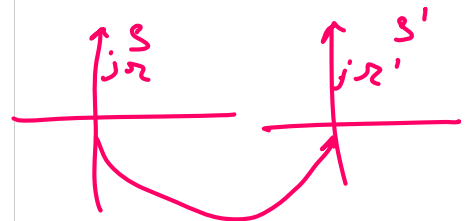
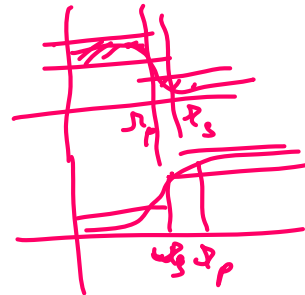


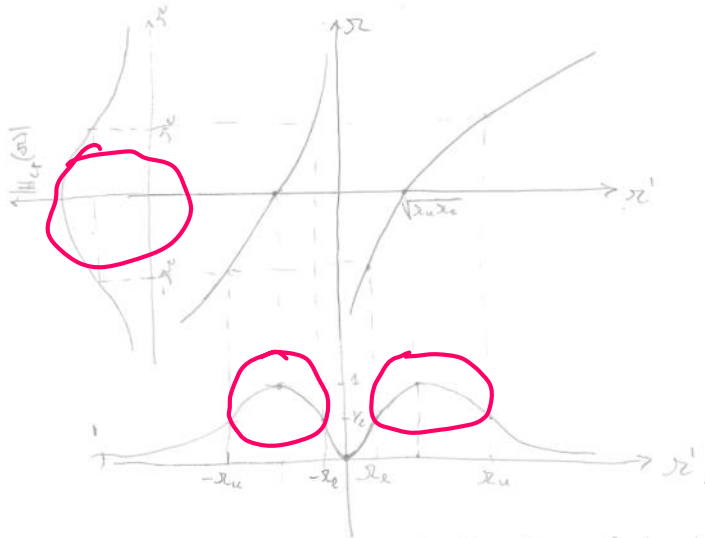
La prima cosa da notare è che per $S' = j\omega'$ abbiamo

$$S = Z_c \frac{(j\omega')^2 + Z_e Z_u}{j\omega' (Z_u - Z_e)} = -Z_c j\omega' \frac{-\omega'^2 + Z_e Z_u}{Z_u - Z_e} = j\omega'$$

Quindi la formula di trasformazione delle frequenze da ω' a ω è

$$\omega = \frac{Z_u}{Z_u - Z_e} \cdot \frac{\omega'^2 - Z_e Z_u}{\omega'} \quad (44)$$





La banda passante si mappa tra σ_c e σ_u e la banda occulta tra 0 e σ_c e tra σ_u e infinito.

Per verificare la stabilità basta analizzare la formula di trasformazione rendendo evidente la parte reale, ovvero per $s' = \alpha' + j\omega'$ abbiamo:

$$s = \frac{\sigma_c}{\sigma_u - \sigma_c} \left[\frac{s' + \frac{\sigma_c \sigma_u}{s'}}{s'} \right] = \frac{\sigma_c}{\sigma_u - \sigma_c} \left[(\alpha' + j\omega') + \sigma_c \sigma_u \frac{(\alpha' - j\omega')}{\alpha'^2 + \omega'^2} \right] =$$

$$= \frac{\sigma_c}{\sigma_u - \sigma_c} \left[\underbrace{\left(1 + \frac{\sigma_c \sigma_u}{\alpha'^2 + \omega'^2} \right)}_{\text{Re}} \alpha' + j \underbrace{\left(1 + \frac{\sigma_c \sigma_u}{\alpha'^2 + \omega'^2} \right)}_{\text{Im}} \omega' \right]$$

Quindi α' conserva il segno di α , ovvero se $H_d(z)$ è stabile, anche $H_d(s)$ è stabile.

Per la trasformazione delle frequenze, notiamo che ogni frequenza in ω corrisponde a due frequenze in ω' . In particolare dal grafico si nota che σ_c si trasforma in σ_u e $-\sigma_c$ e $-\sigma_u$ in σ_c e $-\sigma_u$.

Questo si vede immediatamente invertendo la (***)

TA 5

$$z_c z^2 - (z_u - z_e) z z' - z_c z_e z u = 0$$

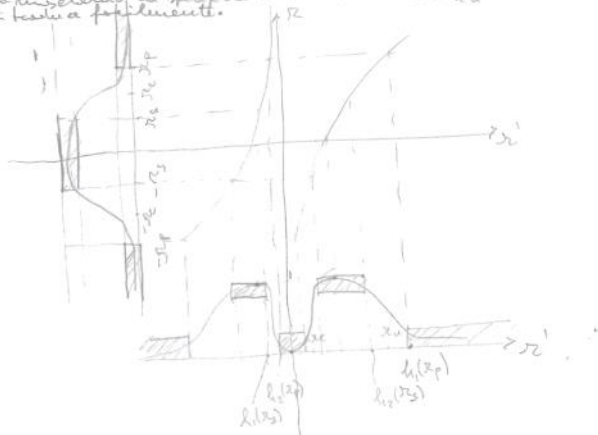
$$z' = \frac{z_c (z_u - z_e) \pm \sqrt{(z_u - z_e)^2 z^2 + 4 z_c^2 z_e z u}}{2 z_c} = \begin{pmatrix} h_+(z) \\ h_-(z) \end{pmatrix}$$

Inoltre per $z = z_c$

$$z' = \frac{z_c (z_u - z_e) \pm \sqrt{(z_u^2 + z_e^2 - 2 z_u z_e + 4 z_c^2 z_e z u) z_c^2}}{2 z_c}$$

$$= \frac{z_c (z_u - z_e) \pm (z_u + z_e) z_c}{2 z_c} = \begin{pmatrix} z_u \\ -z_e \end{pmatrix}$$

Per discutere per $z = z_e$
Usando le soluzioni $h_+(z)$ e $h_-(z)$ $z' = \begin{pmatrix} z_e \\ -z_u \end{pmatrix}$
La singolarità delle superficie nel piano-basso
si trova a fondo.



Le caratteristiche nella banda povera del piano-basso
si sommano in quelle della banda povera del
piano-basso. Quelle della banda ricca del piano-basso nella
caratteristica della due bande ricche del piano-basso.

— 0 —

PASSA-BASSO
fun. di trasferimento $\Rightarrow$ in $[s_e, s_u]$.

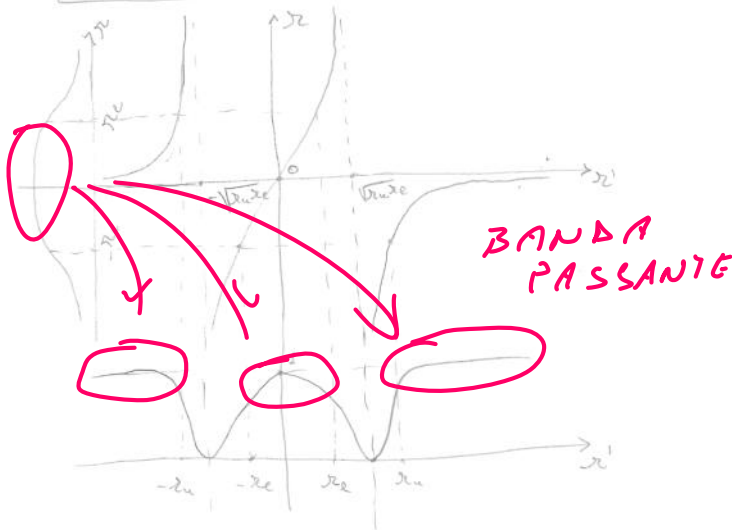
$$S = \pi_c \frac{s'(s_u - s_e)}{s'^2 + s_u s_e}$$

Anche qui per $s' = j\omega$ abbiamo

$$S = \pi_c \frac{j\omega(s_u - s_e)}{-\omega^2 + s_u s_e} \text{ puramente immaginario.}$$

Quindi anche qui l'asse $j\omega$ e l'asse $j\omega'$ sono in relazione diretta e la formula della trasformata delle frequenze è

$$\pi = \pi_c \frac{s'(s_u - s_e)}{-s'^2 + s_u s_e} \quad \left(\frac{\omega}{\omega_c} \right)$$



Per la conservazione della stabilità esaminiamo più nel dettaglio la formula di trasformazione. Per $s' = \alpha' + j\omega'$ TA7

$$s = \pi_c \frac{(\alpha' + j\omega')(\pi_u - \pi_e)}{(\alpha' + j\omega')^2 + \pi_u \pi_e} =$$

$$= \pi_c (\pi_u - \pi_e) \frac{\alpha' + j\omega'}{\alpha'^2 - \omega'^2 + 2j\alpha'\omega' + \pi_u \pi_e} = \frac{\pi_c (\pi_u - \pi_e) (\alpha' + j\omega') (\alpha'^2 - \omega'^2 + \pi_u \pi_e - 2j\alpha'\omega')}{| \cdot |^2}$$

$$= \frac{\pi_c (\pi_u - \pi_e)}{| \cdot |^2} \left[\frac{\alpha'^3 - \omega'^3 + \pi_u \pi_e \alpha' + \omega'^3}{\text{Re}} + j \left(\pi' (\alpha'^2 - \omega'^2 + \pi_u \pi_e) - \alpha' \omega' \right) \right]$$

$\alpha' (\alpha'^2 + \pi_u \pi_e)$
 la parte reale conserva
 il segno op. di il segno dentro di s' e di s sono
 corrispondenti. Quindi un filtro passa-basso
 stabile resta stabile.

Per trasformare le specifiche del passa-basso nelle
 specifiche del filtro a banda passa notiamo
 che ad ogni frequenza del passa-basso corrispondono
 due valori di ω' . Questo si vede invertendo la
 formula di trasformazione alla frequenza $(\omega = 0)$.

$$-\omega'^2 \pi_c + \pi_u \pi_e \pi_c = \pi_c \pi' (\pi_u - \pi_e)$$

$$\pi_c \omega'^2 + \pi_c (\pi_u - \pi_e) \pi' - \pi_u \pi_e \pi_c = 0$$

$$\boxed{\pi' = \frac{-\pi_c (\pi_u - \pi_e) \pm \sqrt{\pi_c^2 (\pi_u - \pi_e)^2 + 4 \pi_u \pi_e \pi_c^2}}{2 \pi_c}}$$

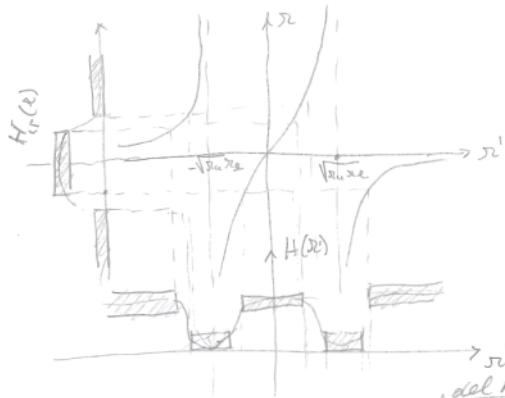
Infatti per $\pi_c = \pi_c$

$$\pi' = \frac{-\pi_c (\pi_u - \pi_e) \pm \sqrt{\pi_c^2 \pi_u^2 + \pi_c^2 \pi_e^2 - 2 \pi_c^2 \pi_u \pi_e + 4 \pi_u \pi_e \pi_c^2}}{2 \pi_c}$$

$$= \frac{-\pi_c (\pi_u - \pi_e) \pm \pi_c (\pi_u + \pi_e)}{2 \pi_c} = \begin{cases} \pi_e \\ -\pi_u \end{cases}$$

Analogamente per $\pi_c = -\pi_c$

$$\pi' = \begin{cases} \pi_u \\ -\pi_e \end{cases}$$



Le specifiche della banda passante ^{del pass-bando} ~~sono le~~
 specifiche delle 2 bande passanti (prima e dopo la
 banda oscura). Le specifiche nella banda oscura del
 pass-bando ~~non sono~~ le caratteristiche della banda
 oscura del filtro finale.

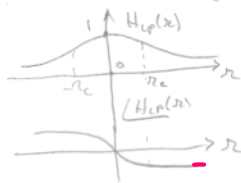
Example

TA 9

Passive elements (Butterworth) $\rightarrow$ PASS BAND $\omega_c \omega_u$

$$H_{LP}(s) = \frac{\omega_c}{s + \omega_c} ; H_{LP}(j\omega) = \frac{\omega_c}{j\omega + \omega_c} = \frac{1}{1 + j\frac{\omega}{\omega_c}}$$

$$|H_{LP}(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^2}} \quad \angle H_{LP}(j\omega) = -\arctan\left(\frac{\omega}{\omega_c}\right)$$



$$\omega_c = 1000 \text{ Hz} \quad \omega_c = 2\pi \cdot 1000 = 6.28 \cdot 10^3 \text{ rad/sec}$$

$$(f_L, f_U) = (5, 6) \text{ kHz} \quad (\omega_L, \omega_U) = (2\pi \cdot 5000, 2\pi \cdot 6000)$$

$$= (15710, 18852) \text{ rad/sec}$$

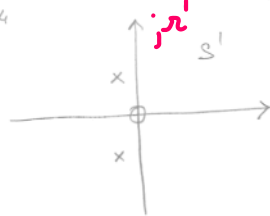
$$H(s') = H_{LP}(s) \Big|_{s = \omega_c s'} = \frac{\omega_c \frac{s'^2 + \omega_c \omega_u}{s'(\omega_u - \omega_c)}}{\omega_c \frac{s'^2 + \omega_c \omega_u}{s'(\omega_u - \omega_c)} + \omega_c}$$

$$= \frac{s'(\omega_u - \omega_c)}{s'^2 + \omega_c \omega_u + s'(\omega_u - \omega_c)} = (\omega_u - \omega_c) \frac{s'}{s'^2 + (\omega_u - \omega_c)s' + \omega_c \omega_u}$$

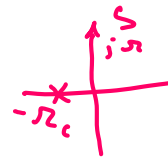
$$\text{poli: } p_{1,2} = \frac{-(\omega_u - \omega_c) \pm \sqrt{(\omega_u - \omega_c)^2 - 4\omega_c \omega_u}}{2} = \frac{\omega_u - \omega_c}{2} \left(-1 \pm \sqrt{1 - \frac{4\omega_c \omega_u}{(\omega_u - \omega_c)^2}} \right)$$

$$= \begin{cases} (-0.3142 + j 3.4211) \cdot 10^4 \\ (-0.3142 - j 3.4211) \cdot 10^4 \end{cases}$$

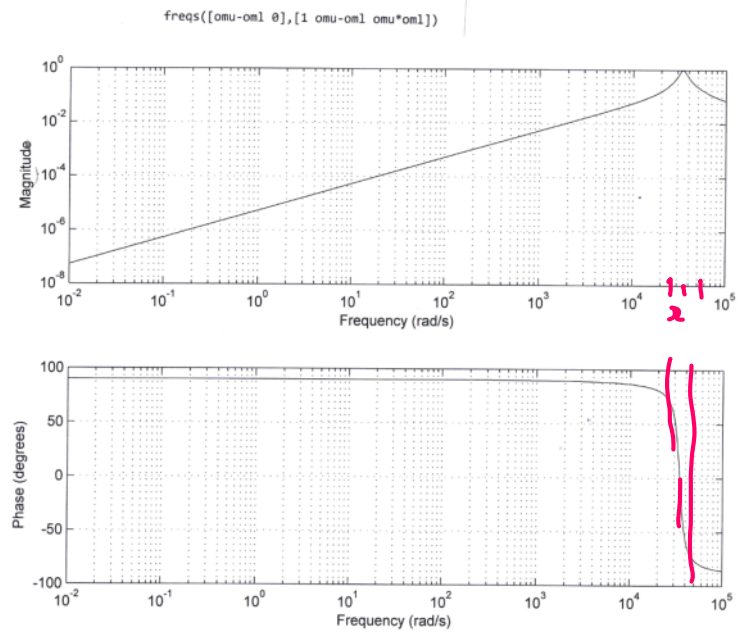
$$z_1 = 0 \quad \text{unstable null anywhere}$$



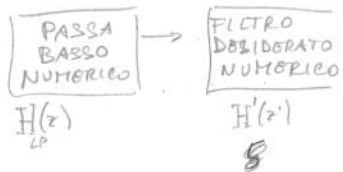
↓ plot on freq.



TA 10



TRASFORMAZIONI



701

8-5. DIGITAL FREQUENCY TRANSFORMATIONS

In this section, digital frequency transformations will be discussed. These transformations can be used to transform a digital low-pass filter into a digital high-pass, band-pass, or band-stop filter. Hence, once a digital low-pass filter is obtained by using the impulse-invariant method, the other type of filters can then be obtained by using digital frequency transformations. This section is the counterpart of Sec. 8-3 for the digital case.

Consider the transformation

$$z = f(z') \quad \text{or} \quad e^{j\omega T} = f(e^{j\omega' T})$$

The function $f(z')$ is required to have the property $|f(e^{j\omega' T})| = 1$ for all ω' . A function with this property will be called a unit function. A unit function maps the unit circle on the z' -plane into the unit circle on the z -plane. For illustration, consider the mapping shown in Fig. 8-17(a). It is a two-to-one mapping. We see that both ω'_{p1} and $-\omega'_{p2}$ on the ω' -axis are mapped into $-\omega_p$ on the ω -axis, both $\omega' = 0$ and $\omega' = \pi/T$ are mapped into $\omega = 0$.

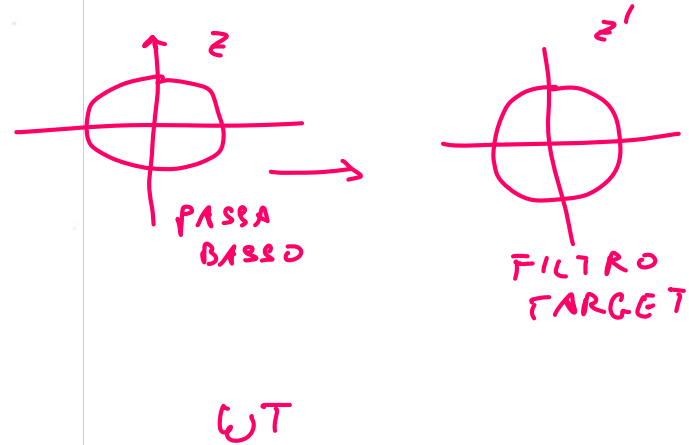
Consider the transfer function $H(z)$ with the amplitude characteristic shown in Fig. 8-17(b). Define

$$H'(z') \triangleq H(z) \Big|_{z=f(z')} = H(f(z'))$$

or

$$H'(e^{j\omega' T}) = H(f(e^{j\omega' T}))$$

The new transfer function $H'(z')$ is obtained from $H(z)$ by the substitution of $z = f(z')$. The amplitude characteristic of $H'(z')$ can be obtained from that of $H(z)$ as shown in Fig. 8-17. For example, the magnitude of $|H'(e^{j\omega' T})|$ at ω'_{p1} and $-\omega'_{p2}$ is equal to the magnitude of $|H(e^{j\omega T})|$ at $-\omega_p$ that is measured as a . By this process



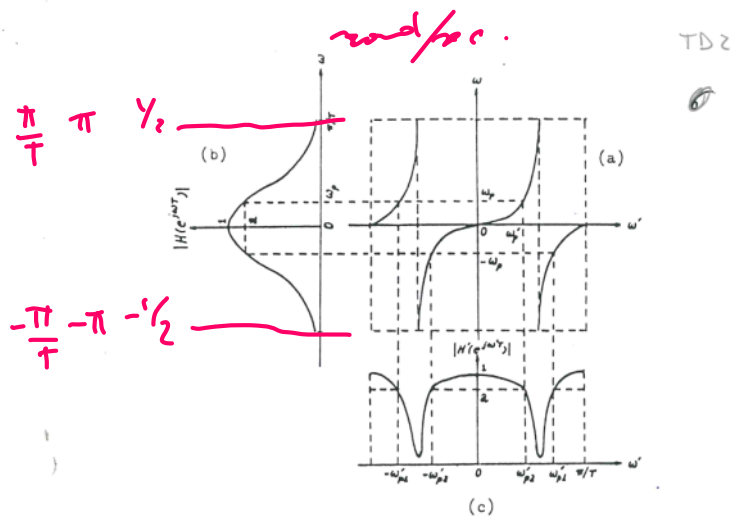


FIG. 8-17. A digital frequency transformation.

the amplitude characteristic of $H'(z')$ can be readily obtained. We see that this transformation converts a low-pass filter into a band-stop filter. This type of transformation is called a *digital frequency transformation*.

Not every function $f(z')$ is a unit function, that is, has the property $|f(e^{j\omega'})| = 1$ for all ω' . We claim that a unit function must be of the form

$$\begin{aligned}
 1 \pm z' \pm z'^{-1} \pm \frac{z' + b}{bz' + 1} \pm \frac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1} \\
 \pm \frac{z'^3 + d_1 z'^2 + d_2 z' + d_3}{d_3 z'^3 + d_2 z'^2 + d_1 z' + 1}
 \end{aligned} \quad (8-46)$$

or $z'^n D(-z)/D(z)$, where n is the degree of the polynomial $D(z)$ with



real coefficients, and b , c_1 , and d_1 are real numbers. We verify this for the last term in (8-46). Indeed, we have

$$\begin{aligned}
 f(e^{j\omega'T}) &= \frac{e^{3j\omega'T} + d_1 e^{2j\omega'T} + d_2 e^{j\omega'T} + d_3}{d_3 e^{3j\omega'T} + d_2 e^{2j\omega'T} + d_1 e^{j\omega'T} + 1} \\
 &= \frac{e^{3j\omega'T} + d_1 e^{2j\omega'T} + d_2 e^{j\omega'T} + d_3}{e^{3j\omega'T} (e^{-3j\omega'T} + d_1 e^{-2j\omega'T} + d_2 e^{-j\omega'T} + d_3)} \\
 &= \frac{(\cos 3\omega'T + d_1 \cos 2\omega'T + d_2 \cos \omega'T + d_3)}{e^{3j\omega'T} (\cos 3\omega'T + d_1 \cos 2\omega'T + d_2 \cos \omega'T + d_3)} \\
 &\quad + j \frac{(\sin 3\omega'T + d_1 \sin 2\omega'T + d_2 \sin \omega'T)}{-j (\sin 3\omega'T + d_1 \sin 2\omega'T + d_2 \sin \omega'T)}
 \end{aligned}$$

Since the magnitudes of the real and imaginary parts of the denominator and numerator are the same, we conclude that $|f(e^{j\omega'T})| = 1$ for all ω' . This shows that the functions in (8-46) are indeed unit functions. (See Problem 8-24.)

With this background, we are ready to discuss the required transformation in the design of various digital filters.

Low-Pass-to-Low-Pass Transformation

Consider the two low-pass filters shown in Fig. 8-18(a); one has a bandwidth ω_p , the other ω'_p . The problem is to find a unit function $z = f(z')$ to achieve this change of bandwidth.

Consider the transformation

$$z = \frac{z' + b}{bz' + 1} = f_L(z') \quad (8-47)$$

or

$$e^{j\omega T} = \frac{e^{j\omega'T} + b}{be^{j\omega'T} + 1}$$

It is easy to verify that f_L maps $\omega' = 0$ into $\omega = 0$, and $\omega' = \pi/T$ into $\omega = \pi/T$. Now if b is chosen so that f_L maps ω'_p into ω_p , then

TD4

Part II: Digital Filters

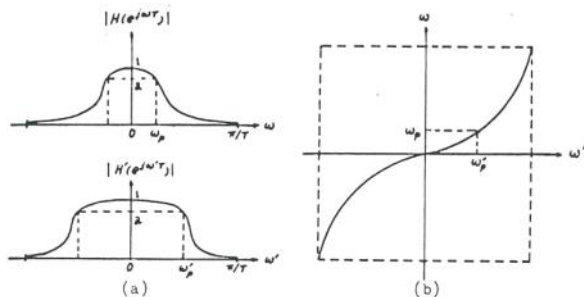


FIG. 8-18. A low-pass-to-low-pass digital frequency transformation.

$f_L(z')$ is the desired transformation. We equate

$$e^{j\omega_p T} = \frac{e^{j\omega'_p T} + b}{be^{j\omega'_p T} + 1}$$

that implies

$$\begin{aligned} b &= \frac{e^{j\omega'_p T} - e^{j\omega_p T}}{e^{j(\omega_p + \omega'_p)T} - 1} \\ &= \frac{e^{j(\omega_p + \omega'_p)T/2} [e^{j(\omega'_p - \omega_p)T/2} - e^{-j(\omega'_p - \omega_p)T/2}]}{e^{j(\omega_p + \omega'_p)T/2} [e^{j(\omega'_p + \omega_p)T/2} - e^{-j(\omega'_p + \omega_p)T/2}]} \\ &= \frac{\sin(\omega'_p T - \omega_p T)/2}{\sin(\omega'_p T + \omega_p T)/2} \end{aligned} \quad (8-48)$$

Hence, using (8-48), the required transformation can be obtained.

The substitution of z by $f_L(z')$ will convert a low-pass filter of bandwidth ω_p into a different low-pass filter of bandwidth ω'_p . We plot in Fig. 8-18(b) the transformation as a function of ω' .

Low-Pass-to-High-Pass Filter

We discuss now the transformation that transforms a low-pass filter into a high-pass filter. Consider the unit function

$$z = f_h(z') = -\frac{z' + b}{bz' + 1} \quad (8-49)$$

or

$$e^{j\omega T} = -\frac{e^{j\omega' T} + b}{be^{j\omega' T} + 1} \quad (8-50)$$

This function transforms $\omega' = 0$ into $\omega = \pi/T$, and $\omega' = \pi/T$ into $\omega = 0$ as shown in Fig. 8-19(a). If $\omega' = \omega'_p$ is transformed into $\omega = \omega_p$, then we have

$$e^{j\omega_p T} = -\frac{e^{j\omega'_p T} + b}{be^{j\omega'_p T} + 1} \quad (8-51)$$

Simple manipulation yields

$$b = -\frac{\cos(\omega'_p T - \omega_p T)/2}{\cos(\omega'_p T + \omega_p T)/2} \quad (8-52)$$

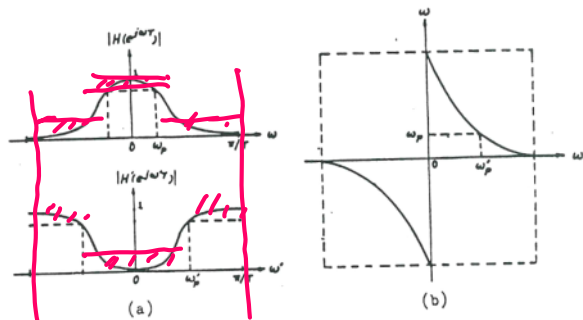
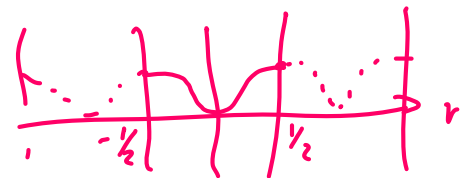
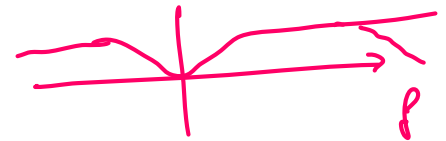


FIG. 8-19. A low-pass-to-high-pass digital frequency transformation.

$$f_{LP}(z) \Big|_{z = \frac{z' + b}{bz' + 1}} = f_H(z')$$



A plot of Eq. (8-50) with b given by (8-52) is shown in Fig. 8-19(b). We note that the b in (8-52) is obtained by transforming $\omega' = \omega'_p$ into $\omega = \omega_p$. If we transform $\omega' = \omega'_p$ into $\omega = -\omega_p$, then we obtain

$$e^{-j\omega_p T} = - \frac{e^{j\omega'_p T} + b}{e^{j\omega'_p T} + 1}$$

which implies

$$b = - \frac{\cos(\omega'_p T + \omega_p T)/2}{\cos(\omega'_p T - \omega_p T)/2} \quad (8-53)$$

The plot of (8-50) as a function of ω' with b given by (8-53) will be in the second and fourth quadrants rather than in the first and third quadrants as for the b given by (8-52). The transformation using either (8-52) or (8-53) will give the same amplitude characteristic but different phase characteristics. Hence, the phase characteristics should be the factor to determine which transform to use.

Low-Pass-to-band-Pass Transformation

The function required in a low-pass-to-band-pass transformation must transform ω'_{p1} into ω_p and ω'_{p2} into $-\omega_p$ as shown in Fig. 8-20(a); therefore, the function must have two parameters or more. Consider

$$z = f_b(z') = - \frac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1} \quad (8-54)$$

or

$$e^{j\omega T} = - \frac{e^{2j\omega'_p T} + c_1 e^{j\omega'_p T} + c_2}{c_2 e^{2j\omega'_p T} + c_1 e^{j\omega'_p T} + 1} \quad (8-55)$$

This function transforms $\omega' = \pi/T$ into $\omega = \pi/T$ and $\omega' = 0$ into $\omega = -\pi/T$. If the function transforms ω'_{p1} into ω_p and ω'_{p2} into $-\omega_p$, then we have

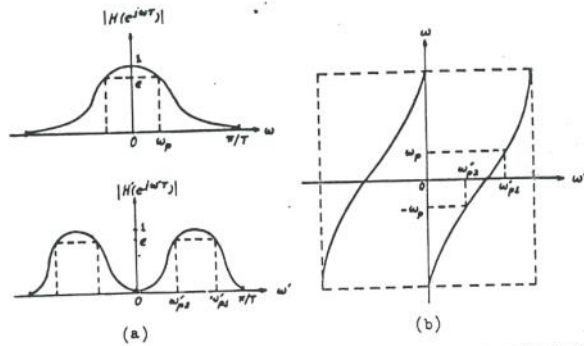


FIG. 8-20. A low-pass-to-band-pass digital frequency transformation.

$$e^{j\omega_p T} = -\frac{e^{2j\omega_p' T} + c_1 e^{j\omega_p' T} + c_2}{c_2 e^{2j\omega_p' T} + c_1 e^{j\omega_p' T} + 1}$$

$$e^{-j\omega_p T} = -\frac{e^{2j\omega_p' T} + c_1 e^{j\omega_p' T} + c_2}{c_2 e^{2j\omega_p' T} + c_1 e^{j\omega_p' T} + 1}$$

These equations can be arranged as

$$c_1 e^{j\omega_p' T} \left(e^{j\omega_p T} + 1 \right) + c_2 \left[e^{j(\omega_p + 2\omega_p') T} + 1 \right] = -e^{2j\omega_p' T} - e^{j\omega_p T}$$

$$c_1 e^{j(\omega_p' - \omega_p) T} \left(1 + e^{j\omega_p T} \right) + c_2 \left[e^{j(-\omega_p + 2\omega_p') T} + 1 \right] = -e^{2j\omega_p' T} - e^{-j\omega_p T}$$

The elimination of c_1 from these equations and simple manipulation yield

$$c_2 = -\frac{e^{j(\omega_p' - \omega_p) T} - e^{j\omega_p' T}}{e^{j\omega_p' T} - e^{j(\omega_p' - \omega_p) T}}$$

From "One Dimensional Digital Signal
Processing" by Alan V. Oppenheim and Ronald W. Schaefer 1979

TABLE 8-3. Digital Frequency Transformations

Low-pass to low-pass	$z = \frac{z' + b}{bz' + 1}$	$b = \frac{\sin(\omega_p' T - \omega_p T)/2}{\sin(\omega_p' T + \omega_p T)/2}$
Low-pass to high-pass	$z = \frac{-(z' + b)}{bz' + 1}$	$b = \frac{-\cos(\omega_p' T - \omega_p T)/2}{\cos(\omega_p' T + \omega_p T)/2}$
Low-pass to band-pass	$z = -\frac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1}$	$h = \frac{\cos(\omega_{p1}' + \omega_{p2}')T/2}{\cos(\omega_{p1}' - \omega_{p2}')T/2}$ $k = \cot(\omega_{p1}' - \omega_{p2}')T/2 \tan \omega_p T/2$ $c_1 = 2h \frac{k}{k+1}, c_2 = \frac{k-1}{k+1}$
Low-pass to band-stop	$z = \frac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1}$	$h = \frac{\cos(\omega_{q1}' + \omega_{q2}')T/2}{\cos(\omega_{q1}' - \omega_{q2}')T/2}$ $k = \tan(\omega_{q1}' - \omega_{q2}')T/2 \tan \omega_q T/2$ $c_1 = -2h \frac{1}{k+1}, c_2 = \frac{1-k}{1+k}$

Let us define

$$k = \frac{(e^{j\omega_{p1}' T} + e^{j\omega_{p2}' T})(1 - e^{-j\omega_p T})}{(e^{j\omega_{p1}' T} - e^{j\omega_{p2}' T})(1 + e^{-j\omega_p T})}$$

$$= \cot \frac{(\omega_{p1}' - \omega_{p2}')T}{2} \tan \frac{\omega_p T}{2} \quad (8-56)$$

Then it can be readily verified that c_2 can be expressed as

$$c_2 = \frac{k-1}{k+1} \quad (8-57)$$

Similarly, the constant c_1 can be solved as

$$c_1 = -2h \frac{k}{k+1} \quad (8-58)$$

where

$$h = \frac{\cos(\omega_{p1}' + \omega_{p2}')T/2}{\cos(\omega_{p1}' - \omega_{p2}')T/2} \quad (8-59)$$

Using Eqs. (8-56) through (8-59), the parameters c_1 and c_2 in (8-54) can be computed. The plot of (8-54) as a function of ω' is shown in Fig. 8-20(b). We see that it is a two-to-one mapping. Note that the point $(\omega_{p1}' + \omega_{p2}')/2$, the midpoint of ω_{p1}' and ω_{p2}' , is not necessarily mapped into $\omega = 0$.

TRASFORMAZIONI EQUIVALENTI

TDS

di PROAKIS KANAKAKIS, "Digital Signal Processing"

8.3 / Frequency Transformations 649

TABLE 8.13 Frequency Transformation for Digital Filters
(Prototype Lowpass Filter Has Cutoff Frequency ω_p)

Type of Transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \frac{z^{-1} - a}{1 - az^{-1}}$	ω_p' = cutoff frequency of new filter $a = \frac{\sin[(\omega_p - \omega_p')/2]}{\sin[(\omega_p + \omega_p')/2]}$
Highpass	$z^{-1} \longrightarrow -\frac{z^{-1} + a}{1 + az^{-1}}$	ω_p' = cutoff frequency of new filter $a = -\frac{\cos[(\omega_p - \omega_p')/2]}{\cos[(\omega_p + \omega_p')/2]}$
Bandpass	$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower cutoff frequency ω_u = upper cutoff frequency $a_1 = \frac{1}{2} \left(\frac{\omega_u}{\omega_l} + \frac{\omega_l}{\omega_u} \right)$ $a_2 = \frac{1}{2} \left(\frac{\omega_u}{\omega_l} - \frac{\omega_l}{\omega_u} \right)$ $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \cot \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$
Bandstop	$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower cutoff frequency ω_u = upper cutoff frequency $a_1 = \frac{1}{2} \left(\frac{\omega_u}{\omega_l} + \frac{\omega_l}{\omega_u} \right)$ $a_2 = \frac{1}{2} \left(\frac{\omega_u}{\omega_l} - \frac{\omega_l}{\omega_u} \right)$ $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \tan \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$

(w) red



ESEMPIO DI PROGETTAZIONE
DI FILTRO NUMERICO

Lezioni di Telecomunicazioni
Prof. FRANCESCO A.N. PASTORI

CORSO DI TRASMISSIONE ED ELABORAZIONE
NUMERICA DEI SEGNALE - SUN
AA. 2014-15

ESEMPIO

E1

Consideriamo un filtro analogico con caratteristiche passa-banda con due poli complessi coniugati a frequenza $f_0 = 3 \text{ [kHz]}$

$$\omega_0 = 2\pi f_0 = 18850 \left[\frac{\text{rad}}{\text{sec}} \right],$$

$$\sigma_0 = 100 \text{ [sec}^{-1}\text{]}$$

$$H_A(s) = \frac{|s_0|^2}{(s - s_0)(s - s_0^*)}$$

$$= \frac{|s_0|^2}{s^2 - s(s_0 + s_0^*) + |s_0|^2}$$

$$= \frac{|s_0|^2}{s^2 - s(2\sigma_0) + |s_0|^2} = \frac{\beta_0^2}{s^2 + 2\sigma_0 s + \beta_0^2}$$

$$s_0 = -\sigma_0 + j\omega_0$$

$$\beta_0^2 = \sigma_0^2 + \omega_0^2$$

$$f_0 = 1.8850 \cdot 10^4$$



La risposta armonica è

$$H_A(\omega) = H_A(s) \Big|_{s=j\omega} = \frac{\beta_0^2}{-\omega^2 + j2\sigma_0\omega + \beta_0^2}$$

Modulo e fase possono essere graficati in MATLAB usando la funzione freqs

```

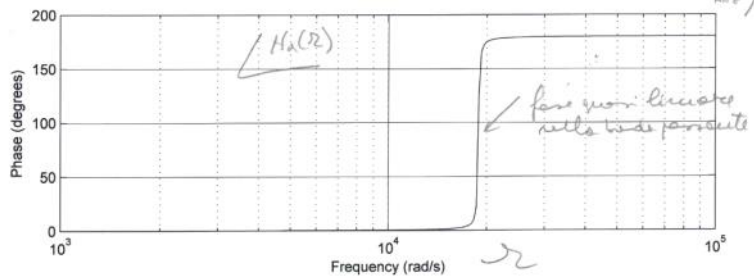
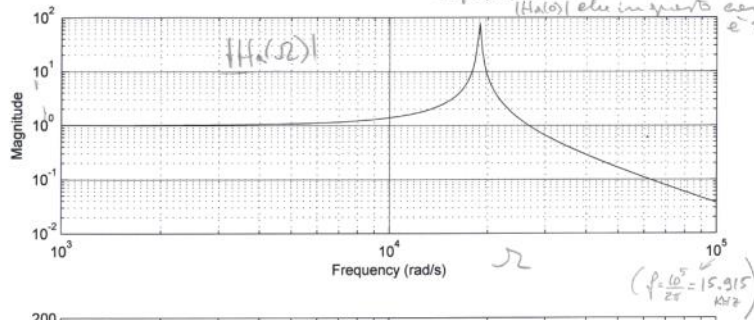
f0 = 3000
omega0 = 2*pi*f0
sigma0 = 100
rho = sqrt(sigma0^2 + omega0^2)
a = [1 -2*sigma0 rho^2]
b = [rho^2]
figure(1)
freqs(b,a)

```

MATLAB
SCRIPT

RISPOSTA DEL
FILTRO ANALOGICO
 $H_a(s)$

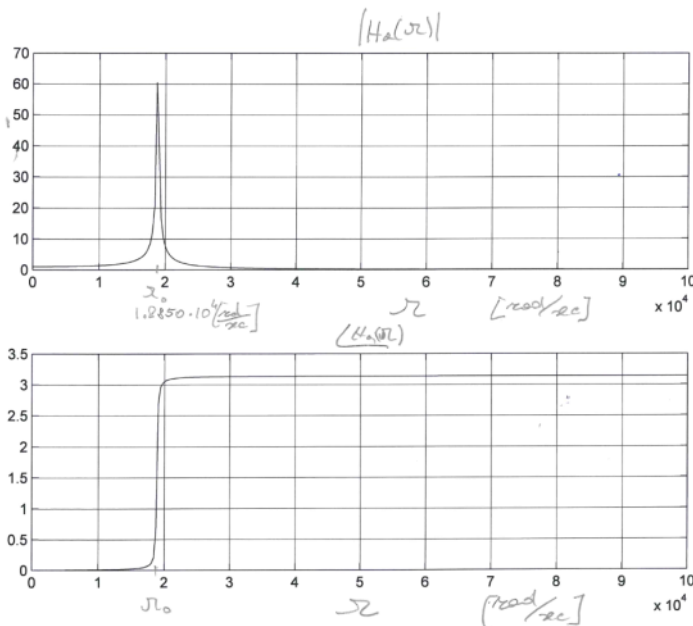
ω è in scala logaritmica
 $|H_a(j\omega)|$ è anch'essa in scala
logaritmica rischiusati a
 $|H_a(j\omega)|$ che in questo caso
è 1



E3

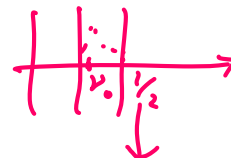
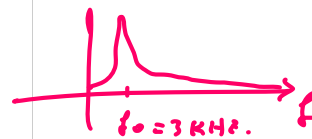
È utile vedere la trama rispetto ad una scala lineare, finché non il predecessore $H_0(0)$ da come è stata costruita $H_0(s)$ o vice.

```
om=linspace(0,100000,300); %generate 300 point on the omega axis from 0 to 10^5
H=freqz(b,a,om); % get the complex response
figure(2)
subplot(2,1,1); plot(om,abs(H)); grid on
subplot(2,1,2); plot(om,phase(H)); grid on
```



La scala logaritmica nelle ampiezze mostra meglio il comportamento nelle code.

Dai grafici si vede lo zero centrato alle frequenze alte gli 8KHz circa.



$$v_0 = \frac{f_0}{f_c} = \frac{3000}{20000} = 0.15$$

$$H(z) = Z \left[\mathcal{L}^{-1} [H_n(s)] \right]_{t=nT}$$



Scelgo una frequenza di campionamento

$$\frac{1}{T} = 20 \text{ KHz} \quad T = 0.5 \cdot 10^{-6} \text{ sec} = 0.5 \mu\text{sec}$$

E4

Per applicare il metodo dell'variazione all'impulso risolviamo $H_n(s)$ alla formula 206 della tabella

$$H_n(s) = \frac{\beta_0^2}{s^2 + 2\zeta_0\beta_0 s + \beta_0^2} = \frac{\beta_0^2}{s^2 + 2\zeta_0\beta_0 s + \beta_0^2} = \frac{\omega_0^2 + \omega_0^2}{(s + \zeta_0\omega_0)^2 + \omega_0^2} \cdot \frac{\omega_0}{\omega_0}$$

$$= \frac{\omega_0^2 + \omega_0^2}{\omega_0} \cdot \frac{\omega_0}{(s + \zeta_0\omega_0)^2 + \omega_0^2} \leftrightarrow \mathcal{L}^{-1} \left[\frac{\omega_0^2 + \omega_0^2}{\omega_0} \cdot \frac{\omega_0}{(s + \zeta_0\omega_0)^2 + \omega_0^2} \right]$$

Il coefficiente $\frac{\omega_0^2 + \omega_0^2}{\omega_0}$ è il coefficiente $\frac{\omega_0^2 + \omega_0^2}{\omega_0}$.

Dopo campionamento e Z-transformata, dalla tabella otteniamo

$$H(z) = \frac{Z \left[\frac{\omega_0^2 + \omega_0^2}{\omega_0} \cdot \frac{\omega_0}{(s + \zeta_0\omega_0)^2 + \omega_0^2} \right]}{z^2 - 2z e^{-\zeta_0\omega_0 T} \cos(\omega_0 T) + e^{-2\zeta_0\omega_0 T}} = \frac{b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Il coefficiente $\frac{\omega_0^2 + \omega_0^2}{\omega_0}$ è il coefficiente $\frac{\omega_0^2 + \omega_0^2}{\omega_0}$.

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n-2]$$

Per valutare le caratteristiche del filtro numerico risolviamo in MATLAB con uno script che valuta modulo e fase di $H(v) = H(z)|_{z=e^{j\omega T}}$, i poli e gli zeri e la risposta impulsiva $h[n] = Z^{-1}[H(z)]$.

risorsa MATLAB con
e fase di $H(\nu) = |H(z)|_{z=e^{j2\pi\nu}}$, i poli e gli zeri e la
risposta impulsiva $h[n] = \mathcal{Z}^{-1}[H(z)]$.

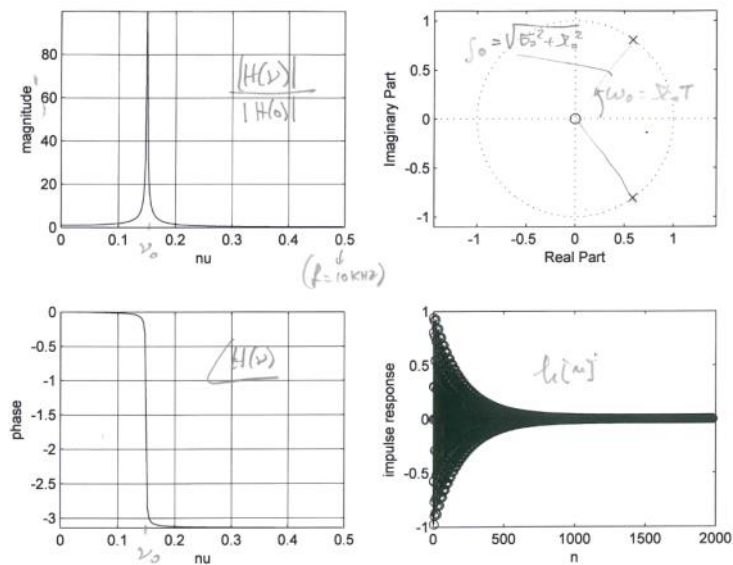
Metodo dell'invarianza all'impulso

```
T=1/20000; %sampling interval
B=[0 exp(-sigma0*T)*sin(omega0*T)]
A=[1 -2*exp(-sigma0*T)*cos(omega0*T) exp(-2*sigma0*T)]
```

```
figure(3)
[H,W]=freqz(B,A,200); %get frequency response (abscissa in rad)
[h,t]=impz(B,A); %get impulse response
subplot(2,2,1)
plot(W/(2*pi),abs(H)/abs(H(1))) %plot magnitude (abscissa in norm freq.; magnitude normalized to the value in 0)
ylabel('magnitude')
xlabel('nu')
axis([0 0.5 0 max(abs(H))])
grid on
subplot(2,2,3)
plot(W/(2*pi),phase(H)) %plot phase (abscissa in norm freq.)
ylabel('phase')
xlabel('nu')
axis([0 0.5 min(phase(H)) max(phase(H))])
grid on
subplot(2,2,2)
%compute and plot z plane
zplane(B,A)
subplot(2,2,4)
stem(t,h) %plot impulse response
ylabel('impulse response')
xlabel('n')
```

(SCRIPT 1)

FILTRO NUMERICO OTTENUTO
CON IL METODO DELL'
INVARIANZA ALL'IMPULSO



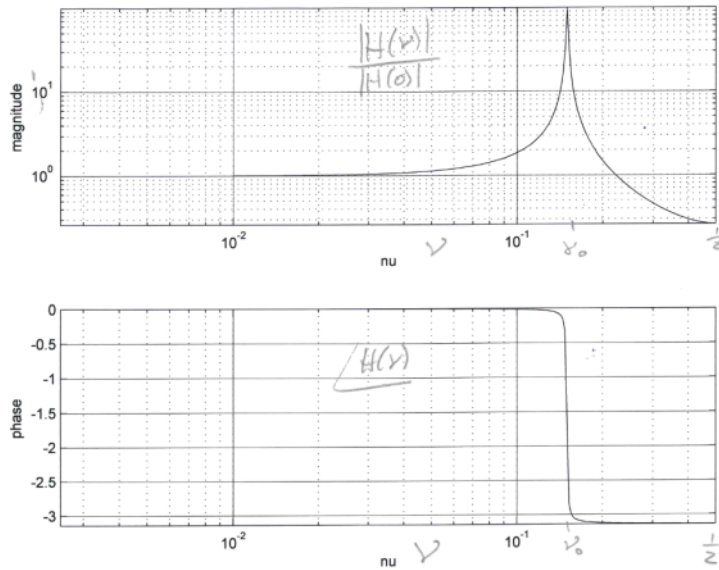
$$\nu_0 = \frac{\omega_0}{2\pi} = 0.15$$

Tubocorrente riferisce il grafico di $|H(\nu)|$ in scala loglog e quello di $\angle H(\nu)$ in semilogx

EC

```
figure(4)
subplot(2,1,1)
loglog(W/(2*pi),abs(H)/abs(H(1))) %plot magnitude (abscissa in norm freq.)
ylabel('magnitude')
xlabel('nu')
axis([0 0.5 0 max(abs(H))])
grid on
subplot(2,1,2)
semilogx(W/(2*pi),phase(H)) %plot phase (abscissa in norm freq.)
ylabel('phase')
xlabel('nu')
axis([0 0.5 min(phase(H)) max(phase(H))])
grid on
```

(script 2)



Per applicare la formula dell'invarianza al gradino
riduciamo $\frac{H(s)}{s}$ alla forma 31 della tabella

$$\frac{H(s)}{s} = \frac{\sigma_0^2 + \omega_0^2}{s[(s + \sigma_0)^2 + \omega_0^2]} \xrightarrow{y^{-1}} \frac{z}{z-1} \frac{z^2 - z e^{-\sigma_0 T} \cos(\omega_0 T - \theta)}{z^2 - 2z e^{-\sigma_0 T} \cos(\omega_0 T) + e^{-2\sigma_0 T}}$$

$$\xrightarrow{z^{-1}} H(z) = 1 - \frac{z-1}{z^2 - 2\beta z + \gamma} \quad \theta = \arg\left(-\frac{\sigma_0}{\omega_0}\right)$$

$$= \frac{z^2 - 2\beta z + \gamma - (z^2 - z\alpha - z + \alpha)}{z^2 - 2\beta z + \gamma} = \frac{z^2 - 2\beta z + \gamma - z^2 + z\alpha + z - \alpha}{z^2 - 2\beta z + \gamma}$$

$$= \frac{(\alpha + 1 - 2\beta)z - \alpha}{z^2 - 2\beta z + \gamma} = (\alpha + 1 - 2\beta) \frac{z - \frac{\alpha}{\alpha + 1 - 2\beta}}{z^2 - 2\beta z + \gamma}$$

$$= \frac{(\alpha + 1 - 2\beta)z^{-1} - \alpha z^{-2}}{1 - 2\beta z^{-1} + \gamma z^{-2}} \quad \begin{matrix} 2 \text{ poli } \left\{ \begin{matrix} z^{-\sigma_0 T} e^{j\omega_0 T} \\ z^{-\sigma_0 T} e^{-j\omega_0 T} \end{matrix} \right. \\ 1 \text{ zero reale } z^{-1} \end{matrix}$$

$$y[n] = 2\beta y[n-1] + \gamma y[n-2] + (\alpha + 1 - 2\beta)x[n] - \alpha x[n-2]$$

%metodo dell'invarianza al gradino

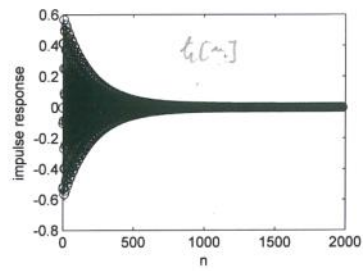
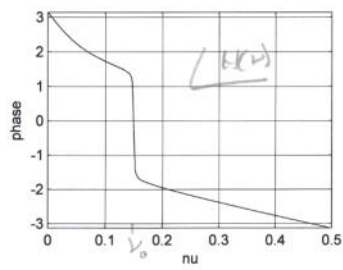
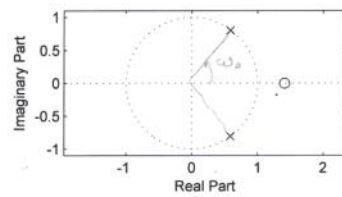
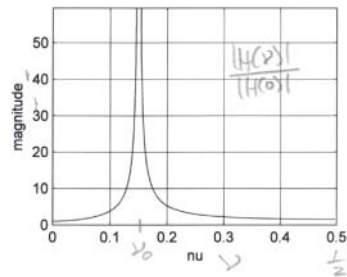
```
theta=atan(-sigma0/omega0)
alpha=exp(-sigma0*T)*sec(theta)*cos(omega0*T - theta)
beta=exp(-sigma0*T)*cos(omega0*T)
gamma=exp(-2*sigma0*T)
A=[1 -2*beta gamma]
B=[0 alpha+1-2*beta -alpha]
```

gli stessi script della pagina precedente sono
utilizzati per ottenere i plot che seguono.

E7

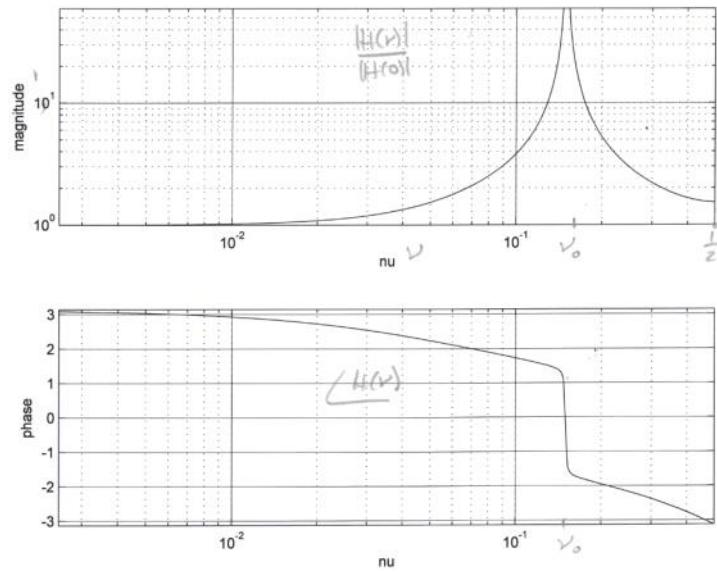
$$H(\tau) = \frac{2}{2.1} \mathcal{F}\left[\sqrt{\frac{H_1(s)}{s}}\right]$$

FILTRO NUMERICO RISULTANTE DALLA TECNICA DELL'INVARIANZA AL BRADINO



eg

(INVARIANZA AL GRADINO)



Per applicare la formula dell'inverso all'ora EIO
 riduciamo $\frac{H_A(s)}{s^2}$ alla forma 107 della tabella

$$\frac{H_A(s)}{s^2} = \frac{\sigma_0^2 + \lambda_0^2}{s^2[(s + \sigma_0)^2 + \lambda_0^2]} = \frac{1}{\sigma_0^2 + \lambda_0^2} \frac{(\sigma_0^2 + \lambda_0^2)^2}{s^2[(s + \sigma_0)^2 + \lambda_0^2]} \xrightarrow{y \rightarrow x}$$

$$\xrightarrow{z} \frac{[(\sigma_0^2 + \lambda_0^2)T + 2\sigma_0]z - 2\sigma_0 z^2}{(z-1)^2} + \frac{2\sigma_0 [z^2 - 2ze^{-\sigma_0 T} \cos \theta \cos(\lambda_0 T - \theta)]}{z^2 - 2ze^{-\sigma_0 T} \cos \lambda_0 T + e^{-2\sigma_0 T}}$$

$$\theta = \tan^{-1} \frac{\lambda_0^2 - \sigma_0^2}{2\sigma_0 \lambda_0}$$

$$H(z) = \frac{1}{\sigma_0^2 + \lambda_0^2} \left\{ \frac{(z-1)^2}{T^2} \frac{[(\sigma_0^2 + \lambda_0^2)T + 2\sigma_0]z - 2\sigma_0 z^2}{(z-1)^2} + \frac{(z-1)^2}{T^2} \frac{2\sigma_0 [z^2 - 2ze^{-\sigma_0 T} \cos \theta \cos(\lambda_0 T - \theta)]}{z^2 - 2ze^{-\sigma_0 T} \cos \lambda_0 T + e^{-2\sigma_0 T}} \right\}$$

$$= \frac{1}{\sigma_0^2 + \lambda_0^2} \left\{ \left(\sigma_0^2 + \lambda_0^2 \right) + \frac{2\sigma_0}{T} z + \frac{2\sigma_0}{T} \frac{(z-1)^2}{z^2 - 2ze^{-\sigma_0 T} \cos \lambda_0 T + e^{-2\sigma_0 T}} \right\}$$

$$= 1 + \frac{2\sigma_0}{T(\sigma_0^2 + \lambda_0^2)} z + \frac{2\sigma_0}{T(\sigma_0^2 + \lambda_0^2)} \frac{(z-1)^2}{z^2 - 2\beta z + \gamma}$$

$$= \frac{[(1+\delta) - \delta z](z^2 - 2\beta z + \gamma) + \delta(z^2 - \alpha z^2 + z - \alpha - z^2 z + 2\alpha z)}{(z^2 - 2\beta z + \gamma)}$$

$$= \frac{[(1+\delta)z^2 - 2\beta(1+\delta)z + (1+\delta)\gamma] + \delta z^2 + 2\delta\beta z - \delta\gamma z + \delta z^2 - \alpha\delta z^2 + \alpha\delta z - 2\delta z^2 + 2\alpha\delta z}{z^2 - 2\beta z + \gamma}$$

$$= \frac{(1+\delta - \alpha\delta)z^2 + (-2\beta(1+\delta) + 2\delta\beta - \delta\gamma + \delta + 2\alpha\delta)z + (1+\delta)\gamma - \alpha\delta}{z^2 - 2\beta z + \gamma}$$

$$= \frac{(1-\delta-\alpha\delta)z^2 + (-2\beta - \delta\gamma + \delta + 2\alpha\delta)z + (1+\delta)\gamma - \alpha\delta}{z^2 - 2\beta z + \gamma}$$

$\leftarrow \text{II}^\circ \text{ordine}$

$\leftarrow \text{II}^\circ \text{ordine}$

$$H(z) = (z \dots) \left[\int_0^1 \left[\frac{H_A(s)}{s^2} \right] \right]$$

$\frac{(z-1)}{Tz}$

b_2

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

E11

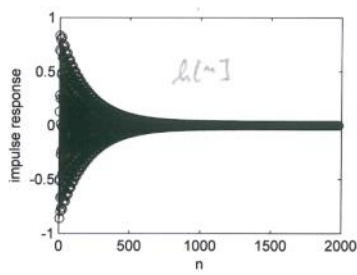
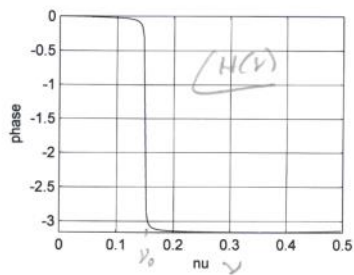
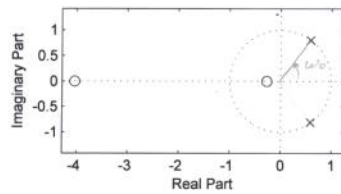
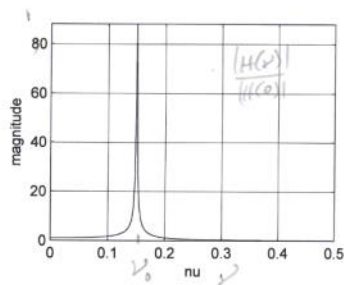
$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$z_{\text{poli}} = \begin{cases} e^{-\sigma_0 T} e^{j\omega_0 T} \\ e^{-\sigma_0 T} e^{-j\omega_0 T} \end{cases}$$

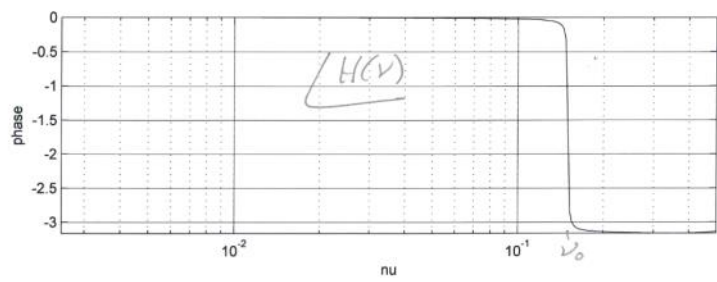
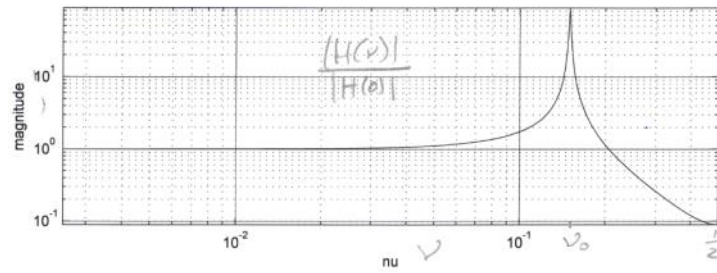
2 poli

% metodo dell'invarianza alla rampa

```
theta=atan((omega0^2 - sigma0^2) / (2*omega0*sigma0))
alpha=exp(-sigma0*T)*sec(theta)*cos(omega0*T - theta)
beta=exp(-sigma0*T)*cos(omega0*T)
gamma=exp(-2*sigma0*T)
delta=(2*sigma0)/(T*(sigma0^2+omega0^2))
A=[1 -2*beta gamma]
B=[1-delta-alpha*delta -2*beta-delta*gamma+2*alpha*delta (1+delta)*gamma-alpha*delta]
```



(invarianza alla scala)



Metodo per approssimazione delle derivate (APPROSSIMAZIONE CAUSALE) E13

$$H(z) = H_A(s) \Big|_{s = \frac{1-z^{-1}}{T}} = \frac{f_0^2}{\left(\frac{z-1}{T^2}\right)^2 + 2\zeta_0 \left(\frac{z-1}{T^2}\right) + f_0^2}$$

$$= \frac{f_0^2}{\frac{z^2+1-2z}{T^2 z^2} + \frac{2\zeta_0}{T} - \frac{2\zeta_0}{T} \frac{1}{z} + f_0^2} = \frac{f_0^2}{\frac{z^2+1-2z+2\zeta_0 T z^2-2\zeta_0 T z+f_0^2 T^2 z^2}{T^2 z^2}}$$

$$= \frac{f_0^2 T^2 z^2}{(1+2\zeta_0 T + f_0^2 T^2) z^2 - 2(1+\zeta_0 T) z + 1}$$

$$= \frac{\frac{f_0^2 T^2}{(1+2\zeta_0 T + f_0^2 T^2)} z^2}{z^2 - \frac{2(1+\zeta_0 T)}{1+2\zeta_0 T + f_0^2 T^2} z + \frac{1}{1+2\zeta_0 T + f_0^2 T^2}} = \frac{b_0}{1+a_1 z^{-1} + a_2 z^{-2}}$$

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] \quad \left. \begin{array}{l} 2 \text{ poli} \\ 2 \text{ zeri in zero} \end{array} \right\}$$

Metodo della approssimazione delle derivate 1 (APP. CAUSALE)

$$A = [1 \quad -2*(1+\text{sigma0}*T)/(1+2*\text{sigma0}*T+\text{rho}^2*T^2) \quad 1/(1+2*\text{sigma0}*T+\text{rho}^2*T^2)]$$

$$B = [(\text{rho}^2*T^2)/(1+2*\text{sigma0}*T+\text{rho}^2*T^2)]$$

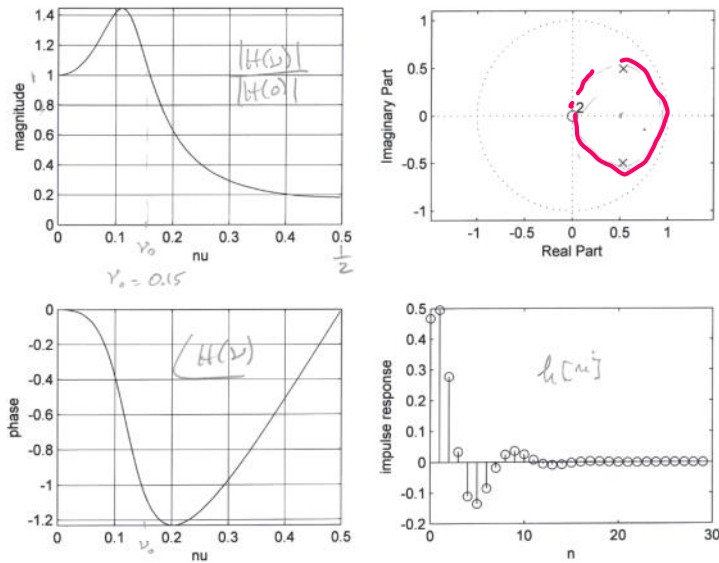
$$A =$$

$$1.0000 \quad -1.0588 \quad 0.5268$$

$$B =$$

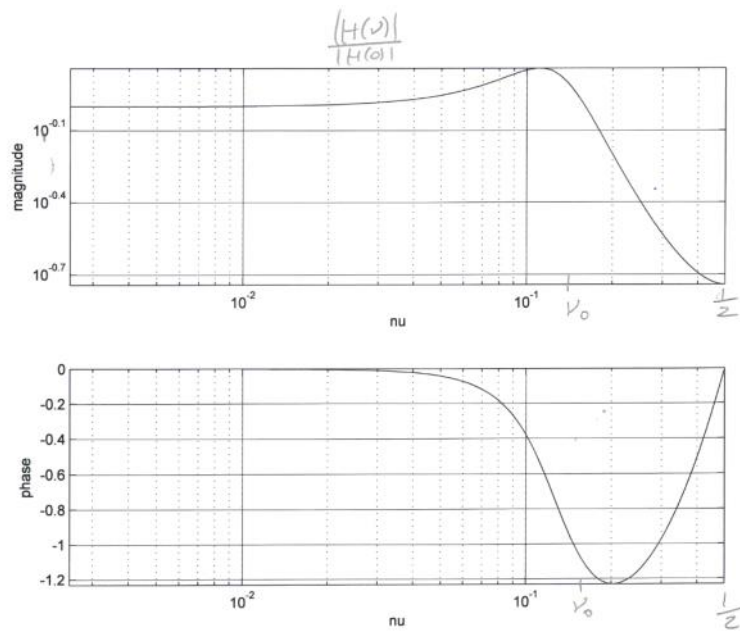
$$0.4679$$

FILTRO NUMERICO DALLA APPROSSIMAZIONE DELLE DERIVATE (APPR. CAUSALE)



Filtro poco selettivo. I poli non possono essere troppo vicini al cerchio unitario a quella frequenza. Si tratta di una trasformazione più adatta a filtri passa-basso o comunque a frequenze di campionamento molto elevate e quindi a caratteristiche paranti a bassa frequenza.

Appr. derivata 1



Metodo per approssimazione delle derivate 2
(App. anticausale) E16

$$H(z) = H_A(s) \Big|_{s=\frac{z-1}{T}} = \frac{\beta_0^2}{\left(\frac{z-1}{T}\right)^2 + 2\zeta_0\left(\frac{z-1}{T}\right) + \beta_0^2}$$

$$= \frac{\beta_0^2}{\frac{z^2+1-zz}{T^2} + \frac{2\zeta_0 z - 2\zeta_0}{T} + \beta_0^2} = \frac{T^2 \beta_0^2}{z^2 + 1 - zz + 2\zeta_0 T z - 2\zeta_0 T + \beta_0^2 T^2}$$

$$= \frac{T^2 \beta_0^2}{\underbrace{z^2}_{a_0} - \underbrace{z(1-\zeta_0 T)}_{a_1} + \underbrace{-2\zeta_0 T + \beta_0^2 T^2}_{a_2}} = \frac{b_0 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

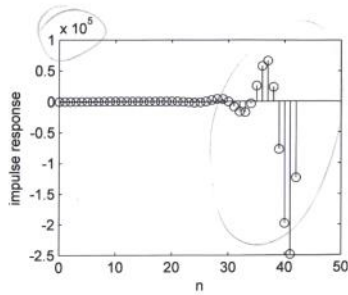
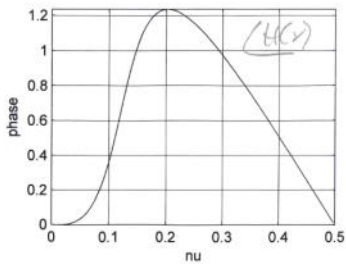
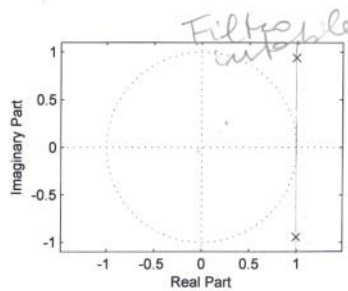
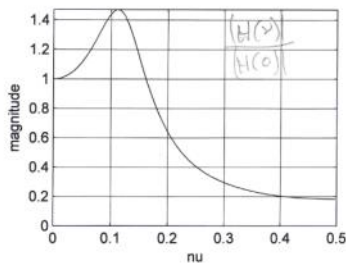
$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n-2]$$

%Metodo della approssimazione delle derivate 2

A=[1 -2*(1-sigma0*T) 1-2*sigma0*T+rho^2*T^2]
B=[0 0 rho*T^2]

A =
1.0000 -1.9900 1.8783
B =
0 0 0.8883

Anche se la risposta in frequenza potrebbe essere superiore, il filtro è instabile (poli fuori dal cerchio unitario)



Se l'obiettivo del filtro numerico è quello di realizzare un passa-banda attorno alla frequenza di 3000 Hz, bisogna adottare il filtro analogico prima della trasformaz.:

PREWARPING

L'obiettivo è ottenere un passa-banda numerico centrato a

$$\omega_0 = \frac{3000}{20000} = 0.15 \quad \text{ovvero a } \omega_0 = 2\pi \nu_0 = 0.9425 \text{ [rad]}$$

(La pulsazione analogica senza warping sarebbe)

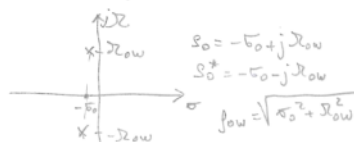
$$\omega_0 = \frac{\omega_s}{T} = 1.8850 \cdot 10^4 \text{ [rad/sec]}$$

Con il warping invece abbiamo

$$\omega_{ow} = \frac{2}{T} \tan \frac{\omega_0}{2} = 2.0381 \cdot 10^4 \text{ [rad/sec]}$$

Quindi il filtro analogico da cui partire è

$$H_a(s) = \frac{\omega_{ow}^2}{s^2 + 2\zeta\omega_{ow}s + \omega_{ow}^2}$$



La frequenza di centro banda è leggermente spostata in avanti.

Applicando la transf. bilineare abbiamo

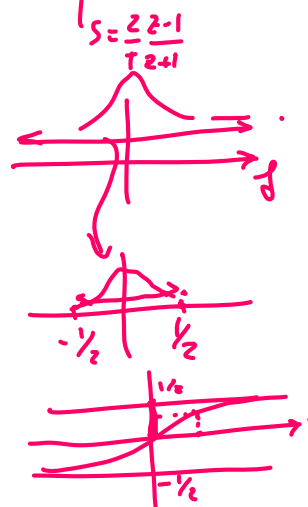
$$H(z) = H_a(s) \Big|_{s = \frac{2}{T} \frac{z-1}{z+1}} = \frac{\omega_{ow}^2}{\frac{4}{T^2} \frac{(z-1)^2}{(z+1)^2} + 2\zeta\omega_{ow} \frac{z-1}{T(z+1)} + \omega_{ow}^2}$$

$$= \frac{\omega_{ow}^2}{\frac{4(z-1)^2 + 4\zeta\omega_{ow}T(z-1) + \omega_{ow}^2T^2(z+1)^2}{T^2(z+1)^2}} =$$

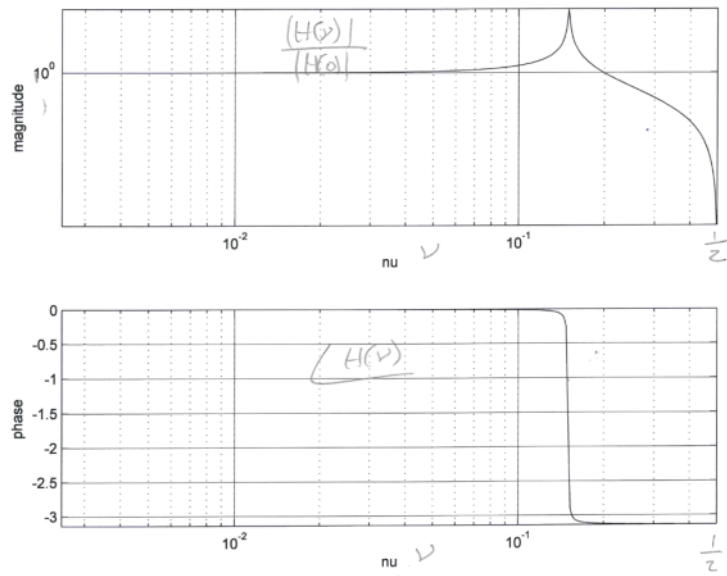
$$= \frac{\omega_{ow}^2T^2z^2 + 2\omega_{ow}^2T^2z + \omega_{ow}^2T^2}{\frac{4z^2 + 4z - 8z + 4\zeta\omega_{ow}Tz - 4\zeta\omega_{ow}T + \omega_{ow}^2T^2z^2 + \omega_{ow}^2T^2z + \omega_{ow}^2T^2}{\omega_{ow}^2T^2z^2 + 2\omega_{ow}^2T^2z + \omega_{ow}^2T^2}}$$

$$= \frac{(4 + 4\zeta\omega_{ow}T + \omega_{ow}^2T^2)z^2 + (-8 + 2\omega_{ow}^2T^2)z + 4 - 4\zeta\omega_{ow}T + \omega_{ow}^2T^2}{(4 + 4\zeta\omega_{ow}T + \omega_{ow}^2T^2)z^2 + (-8 + 2\omega_{ow}^2T^2)z + 4 - 4\zeta\omega_{ow}T + \omega_{ow}^2T^2}$$

$$H(z) = H_a(s) \quad \nu_0 = \frac{f_0}{f_s}$$



FILTRO NUMERICO OTTENUTO MEDIANTE
TRASFORMAZIONE BILINEARE



$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{z^2 + a_1 z + a_2}$$

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

Metodo della trasformazione bilineare

$$\omega_0 = 2/T \cdot \tan(\pi \cdot \nu_0)$$

$$\rho = \sqrt{(\sigma^2 + \omega_0^2)}$$

$$k = 4 \cdot \sigma^2 \cdot \rho^2 \cdot T^2$$

$$A = 1 \cdot (-8 + 2 \cdot \rho^2 \cdot T^2) / k \cdot (4 \cdot \sigma^2 \cdot \rho^2 \cdot T^2)$$

$$B = [(\rho^2 \cdot T^2) / k \cdot (2 \cdot \rho^2 \cdot T^2) / k]$$

$$\omega_0 = 2.0381 \times 10^4$$

$$\rho = 2.0381 \times 10^4$$

$$k = 5.0585$$

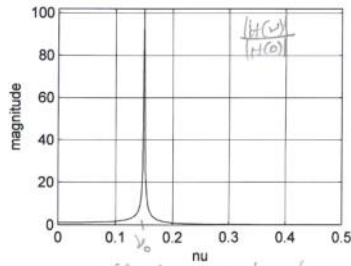
$$A =$$

$$1.0000 \quad -1.1789 \quad 0.9921$$

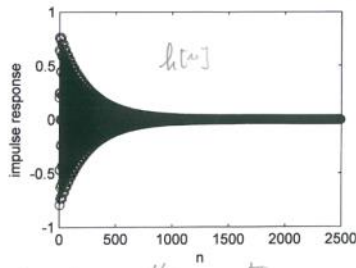
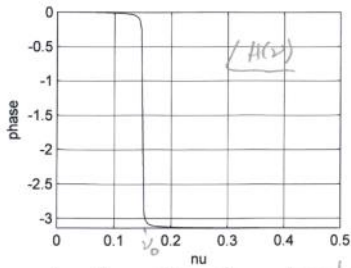
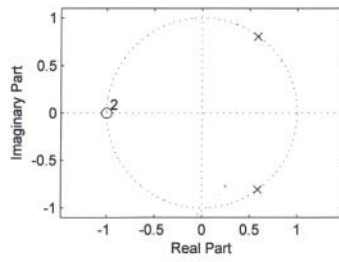
$$B =$$

$$0.2053 \quad 0.4106 \quad 0.2053$$

2 poli stabili
2 poli coincidenti
a $z = -1$



eccellente risposta d'ampiezza.



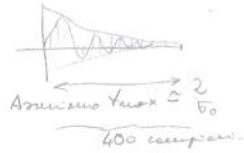
La freq. di centro-banda ν_0 è assolutamente dove desiderato nonostante il warping non lineare dell'axe delle frequenze.

Per completare la raccolta di tecniche di progetto, proponiamo anche un FIR ottenuto come campionamento della risposta impulsiva. Ricordiamo la coppia $H_a(s) \leftrightarrow h_a(t)$

$$H_a(s) = \frac{\beta_0^2}{s^2 + 2\zeta_0 s + \beta_0^2} \longleftrightarrow h_a(t) = \frac{\beta_0^2 + \beta_0^2}{\beta_0} e^{-\zeta_0 t} \sin \beta_0 t$$

$$t_{max} = \frac{2}{\beta_0} = \frac{2}{100} = 0.02 \text{ [sec]}$$

$$N = \left\lceil \frac{t_{max}}{T} \right\rceil = 200$$



$$h[n] = b_n = e^{-\zeta_0 n T} \sin \beta_0 n T \quad n = 0, \dots, N$$

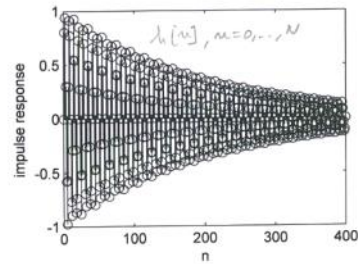
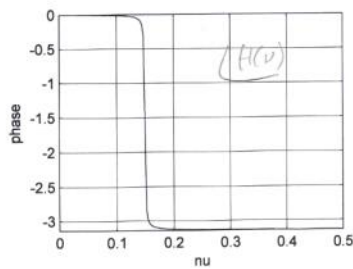
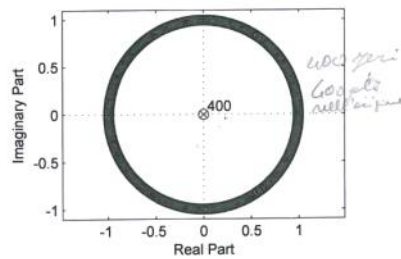
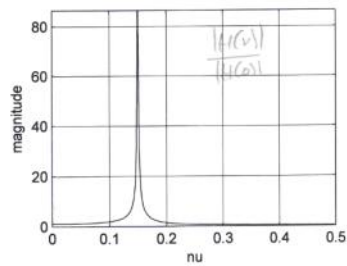
rimuoviamo il coefficiente per avere valori normalizzati

Metodo del FIR dal campionamento e troncamento della risposta impulsiva

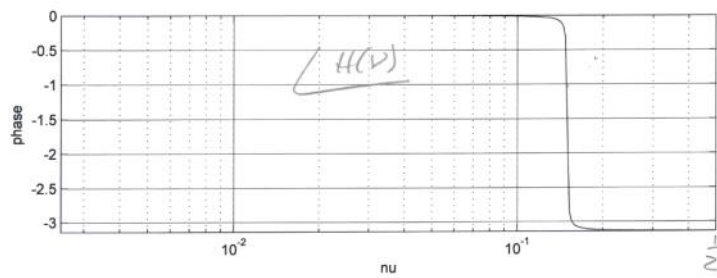
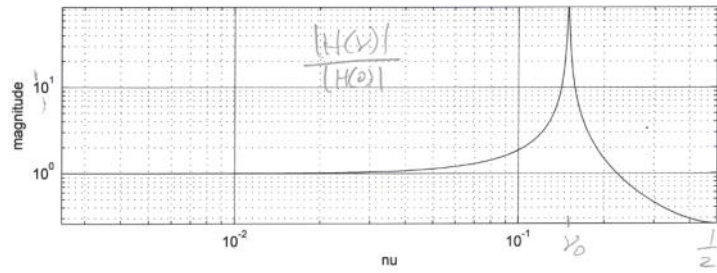
```

tmax=2/sigma0
N=floor(tmax/T)
B=[]
for n=0:N
    B=[B exp(-sigma0*T*n)*sin(omega0*T*n)]
end
A=[1]
    
```

tmax=0.0200
N=400

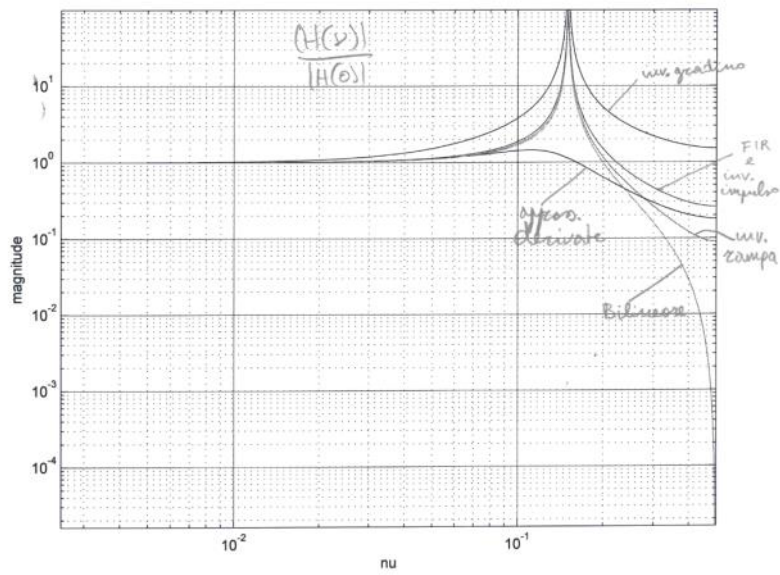


PROGETTO FILTRO FIR
dal campionamento e troncamento a $N=400$
della risposta impulsiva.



CONFRONTO DELLA RISPOSTA DI AMPIEZZA DELLE VARIE TECNICHE

E22



XXXXXXXXXXXXXXXXXXXX

Senza domo

%Metodo dell'invarianza all'impulso

```
T=1/20000; %sampling interval
BII=[0 exp(-sigma0*T)*sin(omega0*T)]
AII=[1 -2*exp(-sigma0*T)*cos(omega0*T) exp(-2*sigma0*T)]
```

%Metodo dell'invarianza al gradino

```
theta=atan(- sigma0 / omega0)
alpha=exp(-sigma0*T)*sec(theta)*cos(omega0*T - theta)
beta=exp(-sigma0*T)*cos(omega0*T)
gamma=exp(-2*sigma0*T)
AIG=[1 -2*beta gamma]
BIG=[0 alpha+1-2*beta -alpha]
```

%Metodo dell'invarianza alla rampa

```
theta=atan((omega0^2 - sigma0^2) / (2*omega0*sigma0))
alpha=exp(-sigma0*T)*sec(theta)*cos(omega0*T - theta)
beta=exp(-sigma0*T)*cos(omega0*T)
gamma=exp(-2*sigma0*T)
delta=(2*sigma0)/(T*(sigma0^2+omega0^2))
AIR=[1 -2*beta gamma]
BIR=[1-delta-alpha*delta -2*beta-delta*gamma+delta+2*alpha*delta (1+delta)*gamma-alpha*delta]
```

%Metodo della approssimazione delle derivate 1

```
AAD=[1 -2*(1+sigma0*T)/(1+2*sigma0*T+rho^2*T^2) 1/(1+2*sigma0*T+rho^2*T^2)]
BAD=[(rho^2*T^2)/(1+2*sigma0*T+rho^2*T^2)]
```

%Metodo della trasformazione bilineare

```
omega0w=(2/T)*tan(pi*nu0)
rho=sqrt(sigma0^2+omega0w^2)
k=4-4*sigma0*T+rho^2*T^2
ATB=[1 (-8+2*rho^2*T^2)/k (4-4*sigma0*T+rho^2*T^2)/k]
BTB=[(rho^2*T^2)/k (2*rho^2*T^2)/k (rho^2*T^2)/k]
```

%Metodo del FIR dal campionamento e troncamento dell risposta impulsiva

```
tmax=2/sigma0
N=floor(tmax/T)
BFI=[]
for n=0:N
BFI=[BFI exp(-sigma0*T*n)*sin(omega0*T*n)]
end
AFI=[1]
```

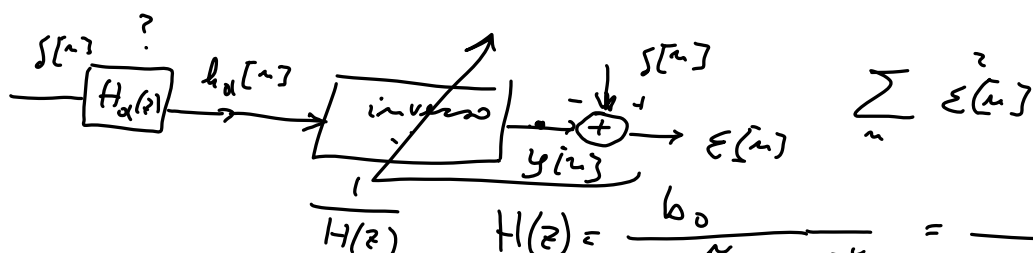
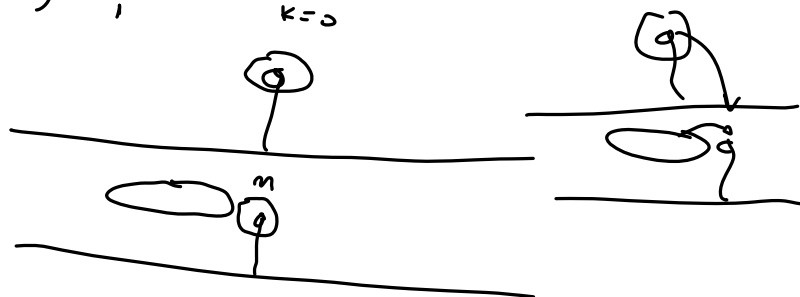
figure(5)

```
[HII W]=freqz(BII,AII,200);
loglog(W/(2*pi),abs(HII)/abs(HII(1)),'b')
ylabel('magnitude')
xlabel('nu')
axis([0 0.5 0 max(abs(HII))])
grid on
hold on
[HIG W]=freqz(BIG,AIG,200);
loglog(W/(2*pi),abs(HIG)/abs(HIG(1)),'r')
hold on
[HIR W]=freqz(BIR,AIR,200);
loglog(W/(2*pi),abs(HIR)/abs(HIR(1)),'g')
hold on
[HAD W]=freqz(BAD,AAD,200);
loglog(W/(2*pi),abs(HAD)/abs(HAD(1)),'k')
hold on
[HTB W]=freqz(BTB,ATB,200);
loglog(W/(2*pi),abs(HTB)/abs(HTB(1)),'c')
hold on
[HFI W]=freqz(BFI,AFI,200);
loglog(W/(2*pi),abs(HFI)/abs(HFI(1)),'k')
```

pagina p



$$y[n] = - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k]$$



$$\frac{1}{H(z)} = \frac{1}{b_0} \left(1 + \sum_{k=1}^{\infty} a_k z^{-k} \right)$$

$$y[n] = \frac{1}{b_0} x[n] + \sum_{k=1}^N \frac{a_k}{b_0} x[n-k]$$

$$y[n] = \frac{1}{b_0} h_d[n] + \sum_{k=1}^N \frac{a_k}{b_0} h_d[n-k]$$

$$y[0] = 1 \quad b_0 = h_0[0]$$

$$\chi^2 = \sum_{k=1}^L (\gamma(x_k) - \delta(x_k))^2$$

$$\sum_{i=1}^L y_i^2$$

favorable a_k $k = 1, \dots, N$

$$= \frac{b_0}{z^{-N} \left(z^N + \underbrace{\sum_{k=1}^N a_k z^{N-k}}_{\downarrow N \text{ pol.}} \right)}$$

N zeros in $z=0$

$h_d(z)$

The diagram shows a horizontal axis labeled z . There are N poles (circles with crosses) on the unit circle at $z=1, 2, 3, 4, 5, \dots$. There are N zeros (circles with dots) at $z=0, 0, 0, 0, 0, \dots$. A vertical line connects the first pole at $z=1$ to the first zero at $z=0$. A double-headed arrow labeled L indicates the length of the system.

$$\mathcal{E} = \sum_{n=1}^L y[n]^2$$

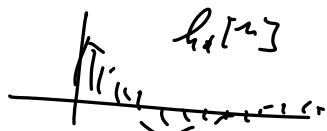
trovare a_k $k=1, \dots, N$
 $\Rightarrow \mathcal{E}$ minimo

$$\mathcal{E} = \sum_{n=1}^L \frac{1}{b_0} \left(b_0 d[n] + \sum_{k=1}^N a_k d[n-k] \right)^2$$

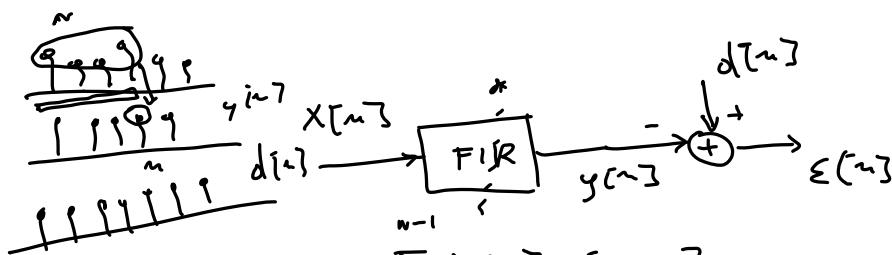
per ottenere il minimo $\mathcal{E}(a_1, a_2, \dots, a_N)$

$$\nabla_a \mathcal{E} = \begin{bmatrix} \frac{\partial \mathcal{E}}{\partial a_1} \\ \frac{\partial \mathcal{E}}{\partial a_2} \\ \vdots \\ \frac{\partial \mathcal{E}}{\partial a_N} \end{bmatrix} = \dots = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}$$

$b_0, a_1, a_2, \dots, a_N$



$$H(z) = \frac{\sum b_k z^{-k}}{1 + \sum a_k z^{-k}}$$

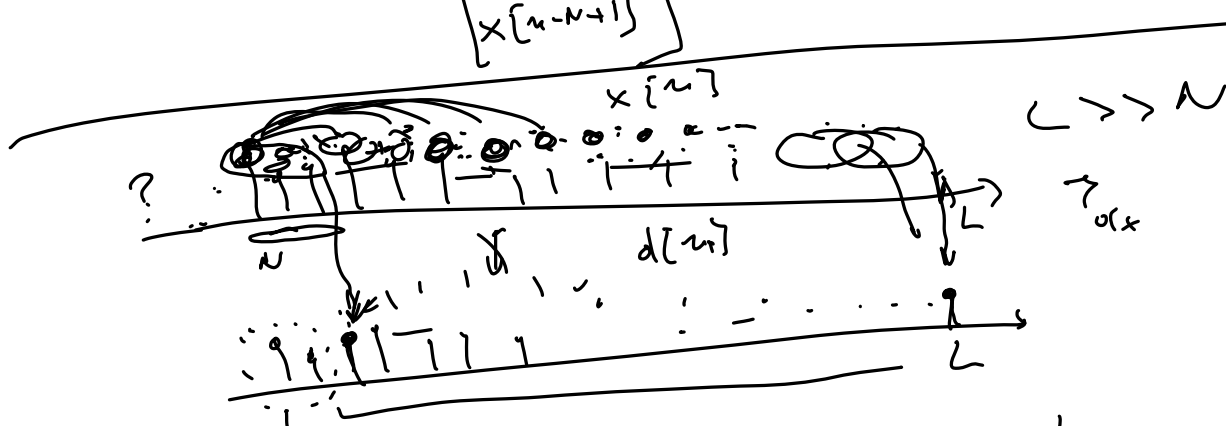


$$y[n] = \sum_{m=0}^{n-1} h[m] x[n-m]$$

$$\underline{x}[n] \triangleq \begin{bmatrix} x[n] \\ x[n-1] \\ x[n-2] \\ \vdots \\ x[n-N+1] \end{bmatrix}$$

$$y[n] = \underline{h}^T \underline{x}[n]$$

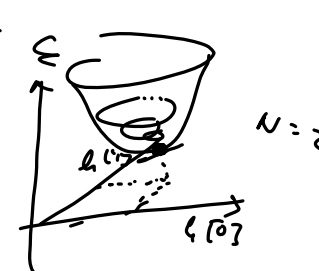
$$\underline{h} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{bmatrix}$$



$$y[n] = \sum_{m=0}^{N-1} h[m] x[n-m] \quad n = N, \dots, L$$

$$\underline{h} = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \quad \mathcal{E}(\underline{h}) = \sum_{n=N}^L (y[n] - d[n])^2$$

$$\underline{h} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{N-1} \end{bmatrix} \quad \mathcal{E}(\underline{h}) = \sum_{n=N}^L (y[n] - d[n])^2$$

$$= \sum_{n=N}^L \left(\sum_{m=0}^{N-1} h[m] x[n-m] - d[n] \right)^2$$


$h_0 = \text{argmin}_{\underline{h}} \mathcal{E}(\underline{h})$

$$\frac{\partial \mathcal{E}}{\partial h[i]} = \sum_{n=N}^L 2 \left(\sum_{m=0}^{N-1} h[m] x[n-m] - d[n] \right) x[n-i]$$

$$= 2 \left(\sum_{m=0}^{N-1} h[m] \sum_{n=N}^L x[n-m] x[n-i] - \sum_{n=N}^L d[n] x[n-i] \right) = 0 \quad i = 0, \dots, N-1$$

$$\sum_{m=0}^{N-1} h[m] \underbrace{\sum_{n=N}^L x[n-m] x[n-i]}_{r_x[i]} = \underbrace{\sum_{n=N}^L d[n] x[n-i]}_{d_x[i]} \quad i = 0, \dots, N-1$$

$$N \begin{pmatrix} R_x \\ \vdots \end{pmatrix} \begin{pmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{pmatrix} = \begin{pmatrix} \vdots \\ d_x \end{pmatrix}$$

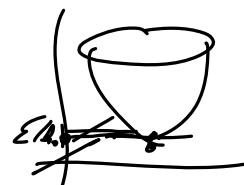
R_x

$$R_x \underline{h}_0 = \underline{d}_x$$

$$\underline{h}_0 = R_x^{-1} \underline{d}_x$$

$$\mathcal{E}(\underline{h})$$

$\mathcal{E}(\underline{h}_0)$



$$y[n] = \underline{h}^T \underline{x}[n] \quad d[n]$$

$$\begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-N+1] \end{bmatrix} \quad \begin{bmatrix} h[0] \\ \vdots \\ h[N-1] \end{bmatrix}$$

$$\mathcal{E}(\underline{h}) = \sum_{n=N}^L (\underline{h}^T \underline{x}[n] - d[n])^2$$

$$= \sum_{n=N}^L (\underline{h}^T \underline{x}[n] - d[n]) (\underline{h}^T \underline{x}[n] - d[n])$$

$\underline{x}[n]^T \underline{h}$

$$\begin{aligned}
 &= \sum_n (\underline{h}_x[n] - d[n]) (\underline{x}[n] - \underline{x}[n]) \\
 &= \sum_n \underline{h}_x^T [n] \underline{x}[n] - 2 d[n] \underline{x}[n] + d[n]^2 \\
 &\underline{h}_x^T \underbrace{\sum_n \underline{x}[n] \underline{x}[n]}_{\substack{\text{matrix} \\ R_x}} - 2 \underbrace{\sum_n d[n] \underline{x}[n]}_{\tau_{dx}} + \sum_n d[n]^2
 \end{aligned}$$

$$\mathcal{E}(\underline{h}) = \underline{h}_x^T R_x \underline{h}_x - 2 \tau_{dx}^T \underline{h}_x + \mathcal{E}_d$$

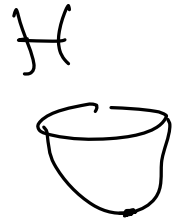
$$\nabla_{\underline{h}} \mathcal{E} = 2 R_x \underline{h}_x - 2 \tau_{dx} = 0$$

$$\begin{aligned}
 \nabla_x \mathcal{E} &= 2 A^T x \\
 b^T x \\
 \nabla_x \mathcal{E} &= b
 \end{aligned}$$

$$H = R_x$$

$$\begin{aligned}
 R_x \underline{h}_0 &= \tau_{dx} \\
 \underline{h}_0 &= R_x^{-1} \tau_{dx}
 \end{aligned}$$

$$R_x^T = R_x$$



$$\mathcal{E}(\underline{h}_0) = \tau_{dx}^T R_x^{-1} R_x R_x^{-1} \tau_{dx} - 2 \tau_{dx}^T R_x^{-1} \tau_{dx} + \mathcal{E}_d$$

$$\mathcal{E}(\underline{h}_0) = \mathcal{E}_d - \tau_{dx}^T R_x^{-1} \tau_{dx} = \mathcal{E}_d - \tau_{dx}^T \underline{h}_0$$

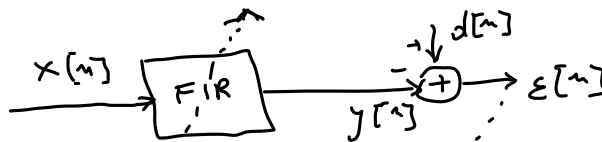
$$R_x = \sum_n \underline{x}[n] \underline{x}[n]^T$$

$$\underline{\xi}^T A \underline{\xi} > 0 \quad \forall \underline{\xi}$$

$$\begin{aligned}
 \underline{\xi}^T R_x \underline{\xi} &= \sum_n \underline{\xi}^T \underline{x}[n] \underline{x}[n]^T \underline{\xi} \\
 &= \sum_n (\underline{\xi}^T \underline{x}[n])^2 > 0
 \end{aligned}$$

$$\begin{aligned}
 &\underline{x}[n] \\
 &\tau_x(k, n)
 \end{aligned}$$

$$(R_x)$$



$\{(x[n], d[n]), n=1, \dots, L\}$
N TRAINING SET

$$\mathcal{E} = \sum_{n=1}^L \epsilon^2[n]$$

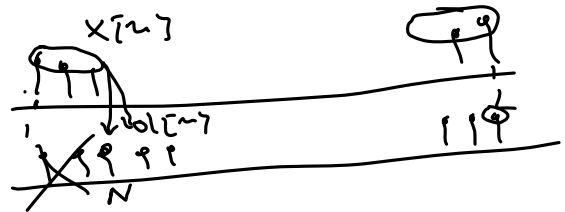
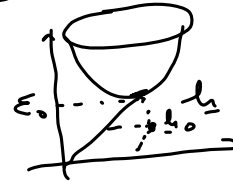
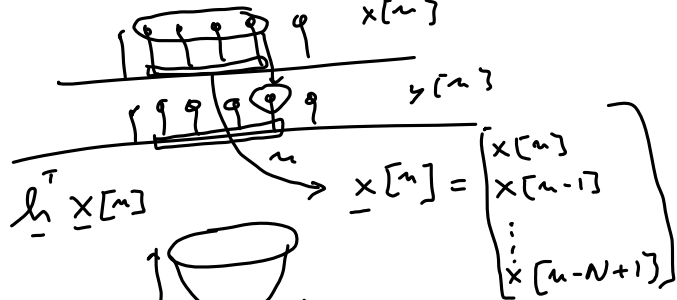
$$y[n] = \sum_{m=0}^{N-1} h[m] x[n-m] = \underline{h}^T \underline{x}[n]$$

$$\mathcal{E}(\underline{h}) = \sum_{n=N}^L (\underline{h}^T \underline{x}[n] - d[n])^2$$

$$\underline{h}_0 = \underline{R}_x^{-1} \underline{z}_{dx}$$

$$\begin{cases} \underline{R}_x = \sum_{n=N}^L \underline{x}[n] \underline{x}[n]^T \\ \underline{z}_{dx} = \sum_{n=N}^L d[n] \underline{x}[n] \end{cases}$$

$$\mathcal{E}_0 = \mathcal{E}(\underline{h}_0) = \sum_{n=N}^L d^2[n] - \underline{z}_{dx}^T \underline{R}_x^{-1} \underline{z}_{dx}$$



$N=3$

$$\underline{R}_x = \sum_n \begin{bmatrix} x[n] \\ x[n-1] \\ x[n-2] \end{bmatrix} \begin{bmatrix} x[n] & x[n-1] & x[n-2] \end{bmatrix}$$

$$= \begin{pmatrix} \sum x^2[n] & \sum x[n]x[n-1] & \sum x[n]x[n-2] \\ \sum x[n-1]x[n] & \sum x^2[n-1] & \sum x[n-1]x[n-2] \\ \sum x[n-2]x[n] & \sum x[n-2]x[n-1] & \sum x^2[n-2] \end{pmatrix}$$

$$= \begin{pmatrix} r_x(0,0) & r_x(0,1) & r_x(0,2) \\ r_x(1,0) & r_x(1,1) & r_x(1,2) \\ r_x(2,0) & r_x(2,1) & r_x(2,2) \end{pmatrix}$$

$$= \begin{pmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(-1) & r_x(0) & r_x(1) \\ r_x(-2) & r_x(-1) & r_x(0) \end{pmatrix}$$

TOEPLITZ



$L \gg 1$

$$\begin{pmatrix} \tau_x(-2) & \tau_x(-1) & \tau_x(0) \end{pmatrix}$$

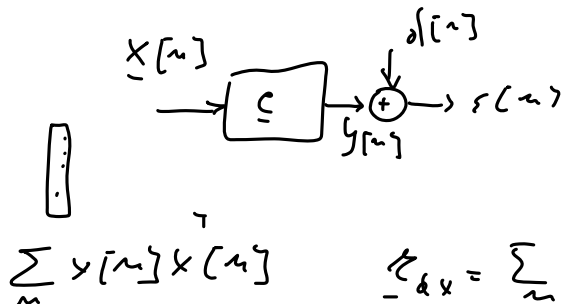
$$\tau_x(1) = \tau_x(-1)$$

n parametri.

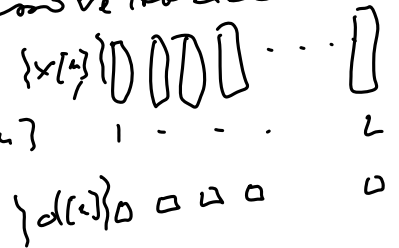
$$= \begin{pmatrix} \tau_x(0) & \tau_x(1) & \tau_x(2) \\ \tau_x(1) & \tau_x(0) & \tau_x(1) \\ \tau_x(2) & \tau_x(1) & \tau_x(0) \end{pmatrix}$$

$$\underline{d}^T | \underline{x} = \sum_n d[n] \underline{x}[n] = \begin{pmatrix} \sum d[n] x[n] \\ \sum d[n] x[n-1] \\ \sum d[n] x[n-2] \end{pmatrix}$$

$$= \begin{bmatrix} \tau_{dx}(0) \\ \tau_{dx}(1) \\ \tau_{dx}(2) \end{bmatrix}$$



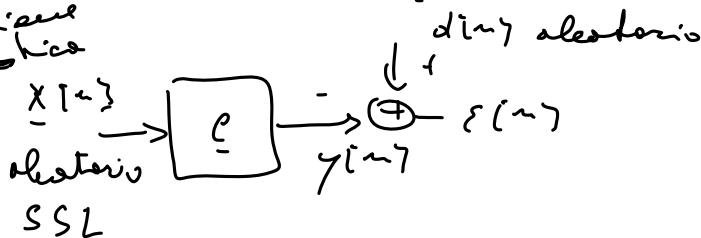
In generale $\underline{x}[n]$
processo vettoriale



$$y[n] = \underline{c}^T \underline{x}[n]$$

$$\underline{c}_0 = \underline{R}_x^{-1} \underline{c}_{dx}$$

Formulazione
stocastica



$$y[n] = \underline{c}^T \underline{x}[n]$$

$$E(\underline{e}) = E[(\underline{c}^T \underline{x}[n] - d[n])^2]$$

$$= \underline{c}^T E[\underline{x}[n] \underline{x}^T[n]] \underline{c} - 2 E[d[n] \underline{x}[n]] + E[d[n]^2]$$

$$= c^T E[x[n] x[n]] c - 2 E[d[n] x[n]] + E[d^2]$$

$$E[x x^T] \quad E[d x] \quad E[d^2]$$

$$c^T R_x c - 2 \sum d x + m_d^2$$

$$\nabla \mathcal{E}(c) = 0$$

$$c_0 = R_x^{-1} \sum d x$$

$$\left\{ \begin{aligned} \hat{R}_x &= \sum_n x[n] x^T[n] \\ \hat{\sum}_{dx} &= \sum_n d[n] x[n] \end{aligned} \right.$$

Formulazione matriciale nel caso:

$$\begin{bmatrix} d[0] \\ d[1] \\ \vdots \\ d[L] \end{bmatrix} = \begin{bmatrix} x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ x[5] & x[4] & x[3] \\ \vdots & \vdots & \vdots \\ x[L-2] & x[L-1] & x[L] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} y[3] \\ y[4] \\ y[5] \\ y[6] \\ \vdots \\ y[L] \end{bmatrix}$$

Toeplitz. A

Sovra-determinato $L-N+1$

$$\boxed{A h = d}$$

$$A h - d = \varepsilon$$

$$\|\varepsilon\|^2 = \varepsilon^T \varepsilon \quad \text{minimizzare } \|\varepsilon\|^2$$

$$h_0 = \underset{h}{\text{argmin}} \varepsilon^T \varepsilon$$

$$\varepsilon^T \varepsilon = (h^T A^T - d^T)(A h - d) = \underbrace{h^T A^T A h}_{\text{box}} - \underbrace{d^T A h}_{\text{box}} - \underbrace{h^T A^T d}_{\text{box}} + \underbrace{d^T d}_{\text{box}}$$

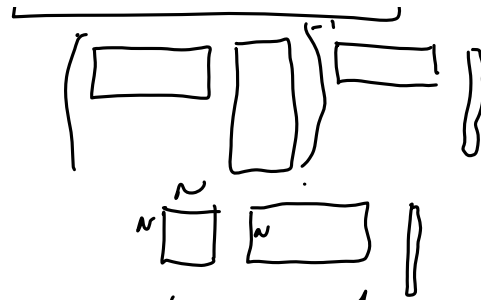
$$\nabla_h \|\varepsilon\|^2 = 2 A^T A h - 2 A^T d = 0$$

$$\boxed{h_0 = (A^T A)^{-1} A^T d}$$

$$\| \text{box} \|$$

$$\| \text{box} \| = 0$$

$$X^T = X$$



$$\begin{bmatrix} A \\ I \end{bmatrix} \begin{bmatrix} l \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix}$$

$$h_0 = A^{-1} d$$

PSEUDO INVERSA $A^\# \triangleq (A^T A)^{-1} A^T$

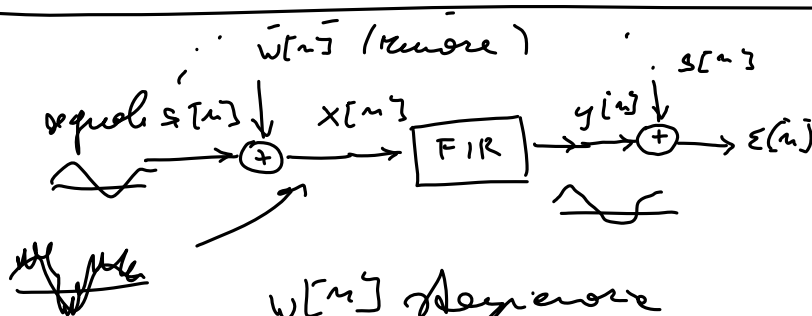
$$\varepsilon_0 = \|\varepsilon_0\|^2 = l^T A^T A l - 2 d^T A l + d^T d \quad \Big|_{l_0 = A^\# d}$$

$$= d^T A (A^T A)^{-1} (A^T A) (A^T A)^{-1} A^T d - 2 d^T A (A^T A)^{-1} A^T d + d^T d$$

$$= d^T A (A^T A)^{-1} A^T d - 2 d^T A (A^T A)^{-1} A^T d + d^T d$$

$$\boxed{\varepsilon_0 = d^T d - d^T A (A^T A)^{-1} A^T d}$$

$$h_0 = (A^T A)^{-1} A^T d$$



$w[n]$ memoria

$s[n]$ memoria

$w[n]$ e $s[n]$ convergono a zero
incrementi.

$\Downarrow$ $x[n]$ stazionario $\Rightarrow R_x$ Toeplitz.

$$R_x = \begin{bmatrix} r_x(0) & r_x(1) & \dots & r_x(N-1) \\ r_x(1) & r_x(0) & \dots & r_x(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(N-1) & \dots & \dots & r_x(0) \end{bmatrix}$$

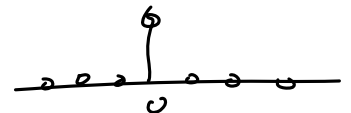
$$\hat{z}_x = E[s] = \begin{bmatrix} E[s(n) x(n)] \\ E[s(n) x(n-1)] \\ \vdots \end{bmatrix}$$

$$\hat{z}_x(l) = E[x(n) x(n-l)] = E[(s(n) + w(n))(s(n-l) + w(n-l))]$$

$$\begin{aligned} r_x(l) &= E[x[n]x[n-l]] = E[(s[n] + w[n])(s[n-l] + w[n-l])] \\ &= E[s[n]s[n-l]] + E[s[n]w[n-l]] + E[w[n]s[n-l]] + E[w[n]w[n-l]] \\ &\quad \underbrace{E[s[n]]E[w[n-l]]}_{=0} \quad \underbrace{E[w[n]s[n-l]]}_{=0} \\ &= R_s(l) + R_w(l) \end{aligned}$$

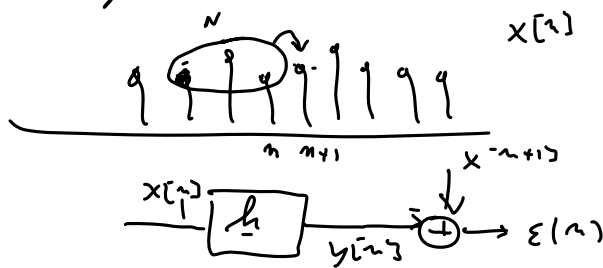
$$\begin{aligned} r_{sx}(m) &= E[s[n]x[n-m]] = E[s[n](s[n-m] + w[n-m])] \\ &= E[s[n]s[n-m]] + E[s[n]w[n-m]] = R_s(m) \end{aligned}$$

Processo bianco $R_w(l) = \sigma_w^2 \delta(l)$



$$\underbrace{\begin{pmatrix} R_s(0) + \sigma_w^2 & R_s(1) & \dots & R_s(N-1) \\ R_s(1) + \sigma_w^2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ R_s(N-1) & \dots & \dots & R_s(0) + \sigma_w^2 \end{pmatrix}}_{h_0} \begin{pmatrix} R_s(0) \\ R_s(1) \\ \vdots \\ R_s(N-1) \end{pmatrix}$$

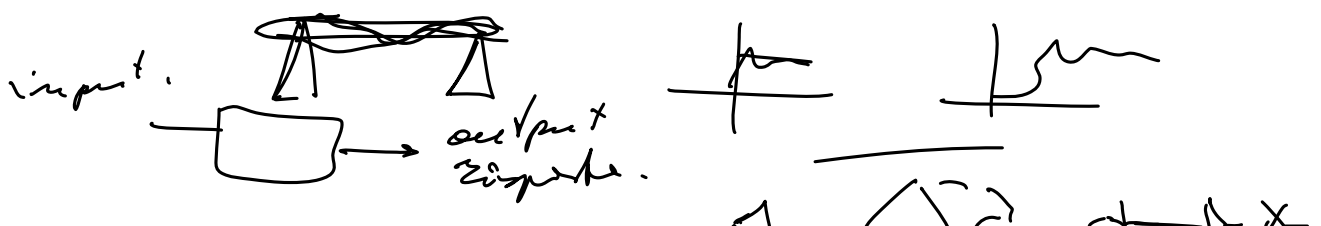
Previsione (lineare)



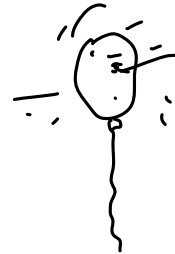
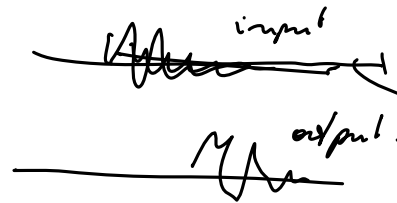
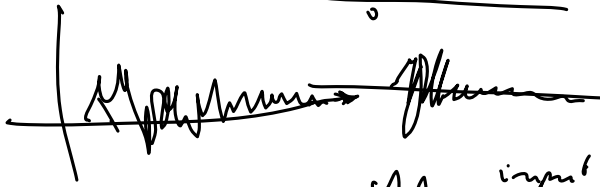
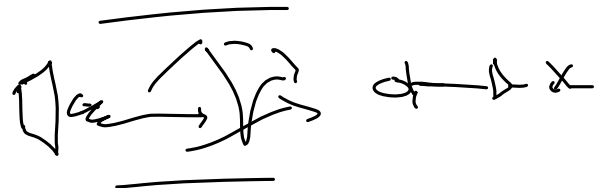
$$\underline{h} = R_x^{-1} \underline{r}_{dx}$$

$$r_{dx}(l) = E[d(n)x[n-l]] = E[x[n+1]x[n-l]] = r_x(l+1)$$

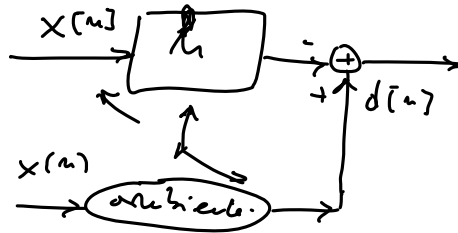
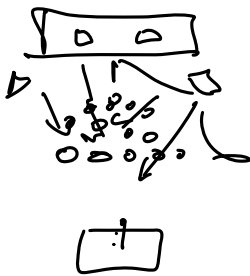
$$\underbrace{\begin{pmatrix} r_x(0) & \dots & r_x(N-1) \\ \vdots & \ddots & \vdots \\ r_x(N-1) & \dots & r_x(0) \end{pmatrix}}_{R_x} \underline{h} = \underbrace{\begin{pmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(N) \end{pmatrix}}_{\underline{r}_{dx}}$$



output
response.

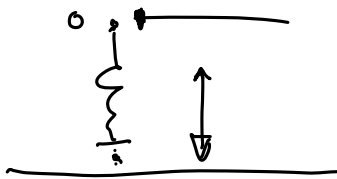


d
 z

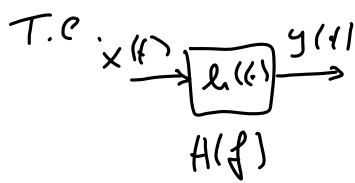


$$\{e_x(e)\} \Rightarrow R_x h_o = \tau_{dx}$$

Handwritten notes



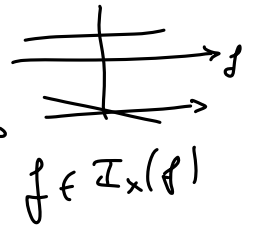
FIR A FASE LINEARE



New sol. $(H(f))|_{f=f_0} = \text{const.} = K$

$\angle H(f) = \text{linear} = -2\pi f t_0$

$$y(t) = Kx(t - t_0)$$



T.D.

$$x[n] \rightarrow h[n] \rightarrow y[n] = (h * x)[n] = \sum_{m=-\infty}^{+\infty} h[m]x[n-m]$$

$$H(\nu) = \sum_{m=-\infty}^{+\infty} h[m]e^{-j2\pi\nu m}$$

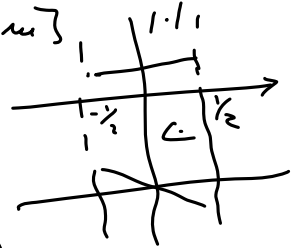
DTFT

$$H(\nu) = K$$

$$\angle H(\nu) = -2\pi\nu m_0$$

$$y[n] = Kx[n - m_0]$$

$$\nu \in I_x(\nu)$$

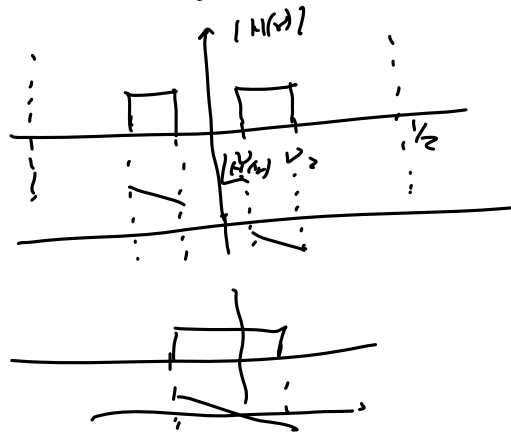


Esempio

Passo banda $f_c = 40 \text{ KHz}$

$$[f_1, f_2] = [3, 5] \text{ KHz}$$

$$[\nu_1, \nu_2] \in \left[\frac{3000}{40000}, \frac{5000}{40000} \right] = \left[\frac{3}{40}, \frac{5}{40} \right]$$

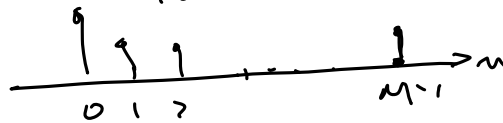


DFT

DTFT

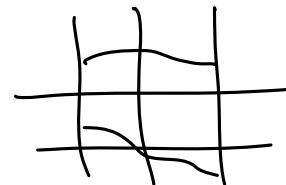
FIR A FASE LINEARE

$$h[n]$$



$$H(\nu) = \sum_{m=0}^{M-1} h[m]e^{-j2\pi\nu m}$$

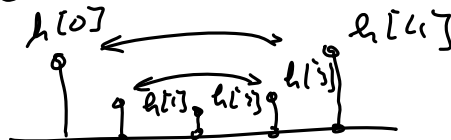
$$y[n] = \sum_{m=0}^{M-1} h[m]x[n-m]$$



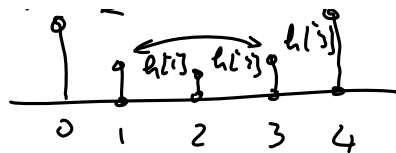
FIR SIMMETRICO



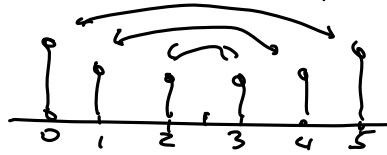
$$M=5$$



11-0



$N=6$



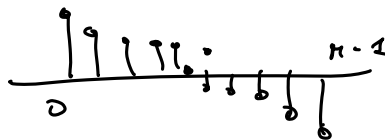
$N=5$

$$\begin{aligned}
 H(\nu) &= h[0] + h[1] e^{-j2\pi\nu} + h[2] e^{-j2\pi\nu 2} + h[3] e^{-j2\pi\nu 3} + h[4] e^{-j2\pi\nu 4} \\
 &= e^{-j2\pi\nu 2} \left(h[0] e^{j2\pi\nu 2} + h[1] e^{j2\pi\nu} + h[2] + h[3] e^{-j2\pi\nu} + h[4] e^{-j2\pi\nu 2} \right) \\
 &= e^{-j2\pi\nu 2} \left(h[0] 2 \cos 2\pi\nu 2 + 2h[1] \cos 2\pi\nu + h[2] \right) \\
 &\quad H_2(\nu)
 \end{aligned}$$

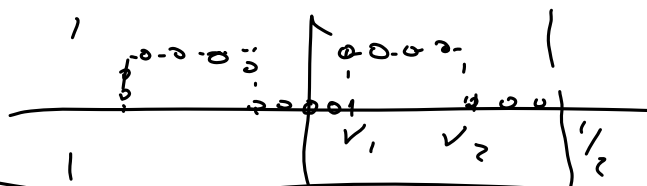
$$\angle H(\nu) = -2\pi\nu 2 + \angle H_2(\nu)$$

Nella banda passante $H_2(\nu)$ è positivo $\Rightarrow$ fase lineare.

FIR ANTISIMMETRICI

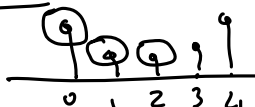


$\Rightarrow$ Fase lineare.

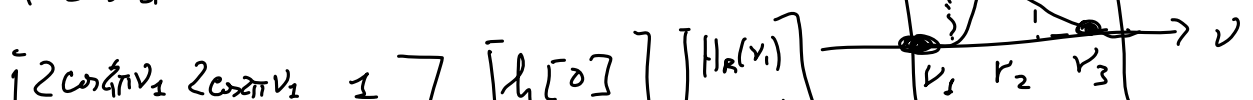


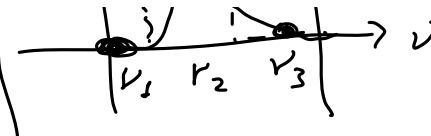
$$N=5 \quad H(\nu) = e^{-j2\pi\nu 2} \left(2h[0] \cos 2\pi\nu + 2h[1] \cos 2\pi\nu + h[2] \right)$$

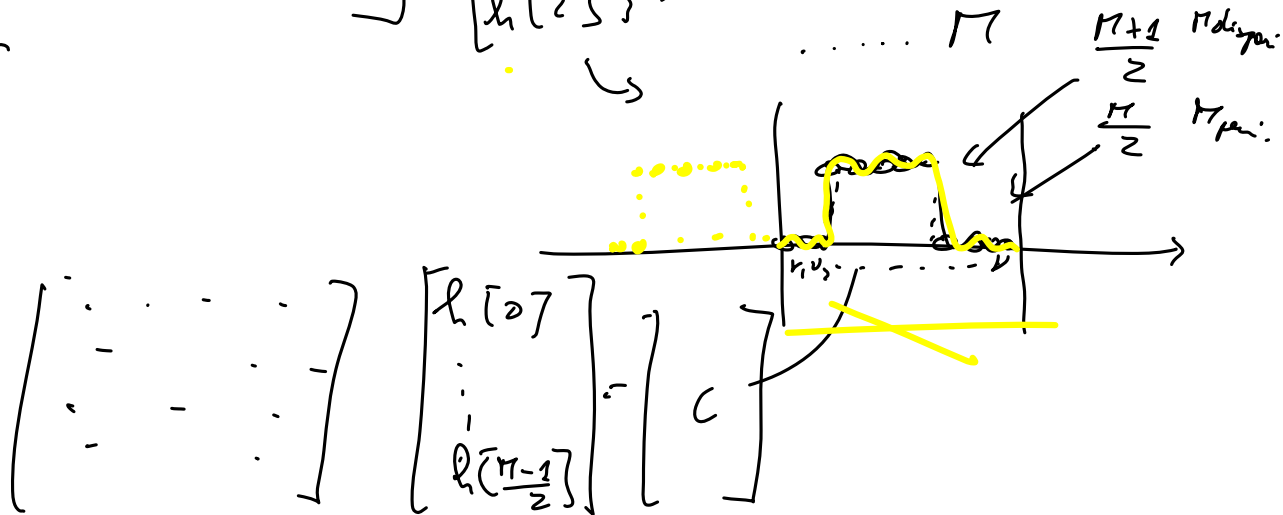
SIMMETRICI



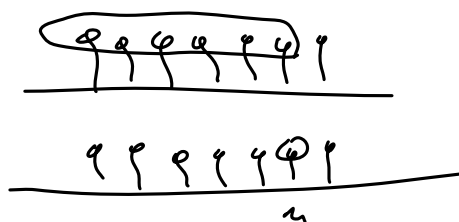
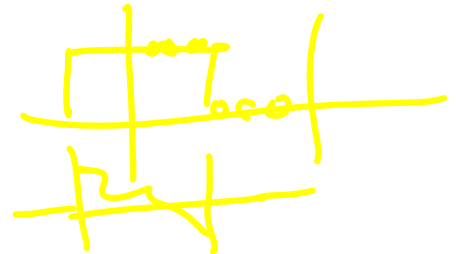
$H_R(\nu)$



$$\begin{bmatrix} 2\cos 4\pi v_1 & 2\cos 2\pi v_1 & 1 \\ 2\cos 4\pi v_2 & 2\cos 2\pi v_2 & 1 \\ 2\cos 4\pi v_3 & 2\cos 2\pi v_3 & 1 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} H_R(v_1) \\ H_R(v_2) \\ H_R(v_3) \end{bmatrix}$$




$$y[n] = \sum_{m=0}^{n-1} h[m] x[n-m]$$



40000

$$\frac{1}{4 \cdot 10^4}$$

$$0.25 \cdot 10^{-4} = 25 \cdot 10^{-6}$$

FILTRI

Da prot. analogici $\rightarrow H(z)$

T. invariante all'impulso
 " al gradiente
 " alla risonanza

Appross. osservate.
 T. Bilinear.

Dallo z-punto complesso $\Rightarrow H(z)$

limiti

Parte
 All-pole

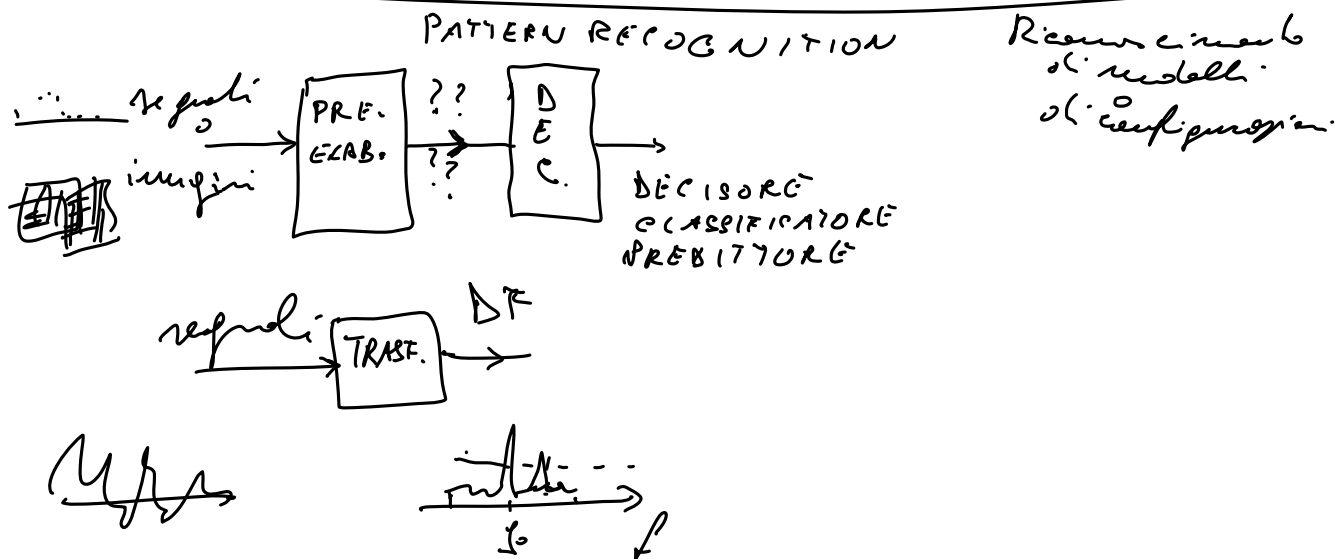
Toxide
All-pole
Poli-zeri
Chouk.

(F. d. Wiener)

Form. minor gradat.
" " Noticulae
Form. distinctae

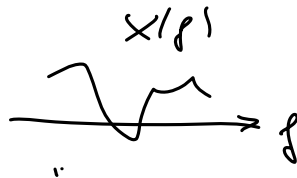
FIR a fase lineare $\Rightarrow$ componenti della risposta sono

TdS { $\begin{matrix} \text{FIR} & \rightarrow & \text{Tecniche della finestra o} \\ & & \text{della Serie di Fourier.} \\ \text{IIR} & \rightarrow & \text{Progettamento pol.-zeri.} \end{matrix}$



Traf. de Février (Projet de décret)

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$



$x(t)$ process elettrico Stazionario

$$R_x(z) \leftrightarrow \tilde{P}_x(l)$$

$$E[X(f)X^*(f-\Delta)] = E \left[\int_{-\infty}^{+\infty} x(t_1) e^{-j2\pi f t_1} dt_1 \int_{-\infty}^{+\infty} x^*(t_2) e^{+j2\pi(f-\Delta)t_2} dt_2 \right]$$

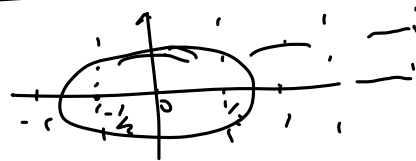
$$\begin{aligned}
 E[X(f)X^*(f-\Delta)] &= E\left[\int_{-\infty}^{+\infty} x(t_1)e^{-j2\pi f t_1} dt_1 \int_{-\infty}^{+\infty} x^*(t_2)e^{+j2\pi(f-\Delta)t_2} dt_2\right] \\
 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[x(t_1)x^*(t_2)] e^{-j2\pi f(t_1-t_2)} e^{-j2\pi \Delta t_2} dt_1 dt_2 \\
 &\quad R_x(t_1-t_2) \quad \xi = t_1 - t_2 \quad ; \quad t_2 = t \\
 &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} R_x(\xi) e^{-j2\pi f \xi} d\xi \right) e^{-j2\pi \Delta t} dt \\
 &= \int_{-\infty}^{+\infty} P_x(f) e^{-j2\pi \Delta t} dt = P_x(f) \int_{-\infty}^{+\infty} e^{-j2\pi \Delta t} dt \\
 &= P_x(f) \delta(\Delta)
 \end{aligned}$$

$$E[X(f)X^*(f-\Delta)] = P_x(f) \delta(\Delta)$$

correlation, Fourier $\Rightarrow$ independent:

$x(t)$ $\xrightarrow{\text{Fourier}}$ $X(f)$ independent
 time domain $\rightarrow$ frequency domain

$$x[n] \xrightarrow{\text{DTFT}} X(\nu) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi \nu n}$$



glog. SSC

$$\text{Autocor. } R_x[m] = E[X[n]X^*[n-m]]$$

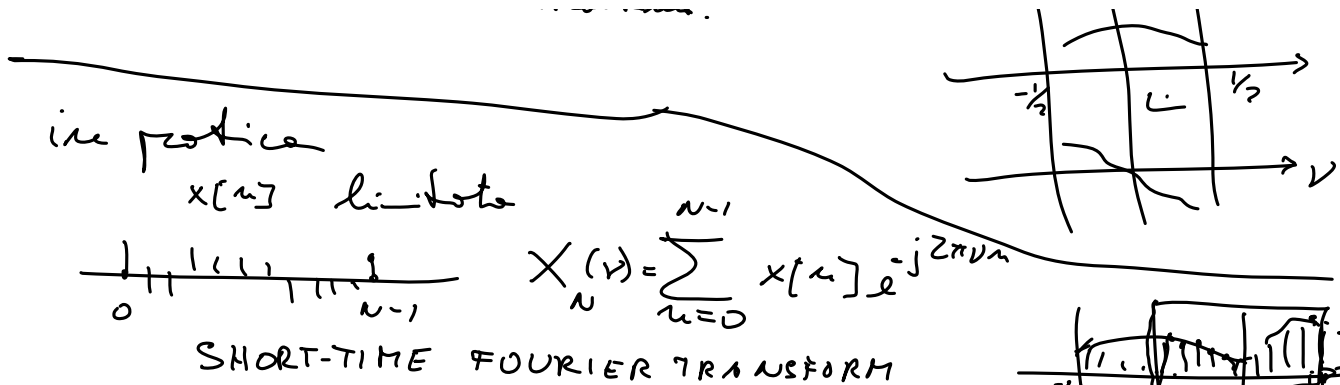
$$\text{Spectrum} \downarrow \text{Power} \quad P_x(\nu) = \sum_{m=-\infty}^{+\infty} R_x[m] e^{-j2\pi \nu m}$$

$$\begin{aligned}
 E[X(\nu)X^*(\nu-\Delta)] \\
 &= P_x(\nu) \sum_{k=-\infty}^{+\infty} \delta(\Delta-k) \\
 &\text{deconvolution}
 \end{aligned}$$

$$x[n] \xrightarrow{\text{DTFT}} X(\nu) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi \nu n}$$

$\uparrow$
 ν continuous.



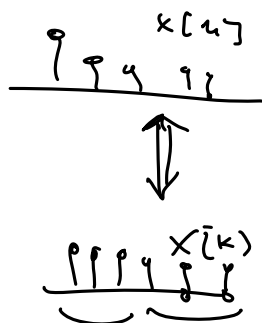


CAMPIONI IN FREQUENZA.

$$\nu = \frac{K}{N} \quad K=0, \dots, N-1$$

$$X(K) = X_N(\nu) \Big|_{\nu = \frac{K}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} K n} \quad K=0, \dots, N-1$$

DFT



$$X[K] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} K n} \quad K=0, \dots, N-1$$

DFT

DISCRETE FOURIER TRANSFORM

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] e^{j\frac{2\pi}{N} K n} \quad n=0, \dots, N-1$$

IDFT

FFT Algoritmo veloce e relativo alla DFT

$$\begin{bmatrix} 1 \\ e^{-j\frac{2\pi}{N} K n} \\ \vdots \\ e^{-j\frac{2\pi}{N} K (N-1)n} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

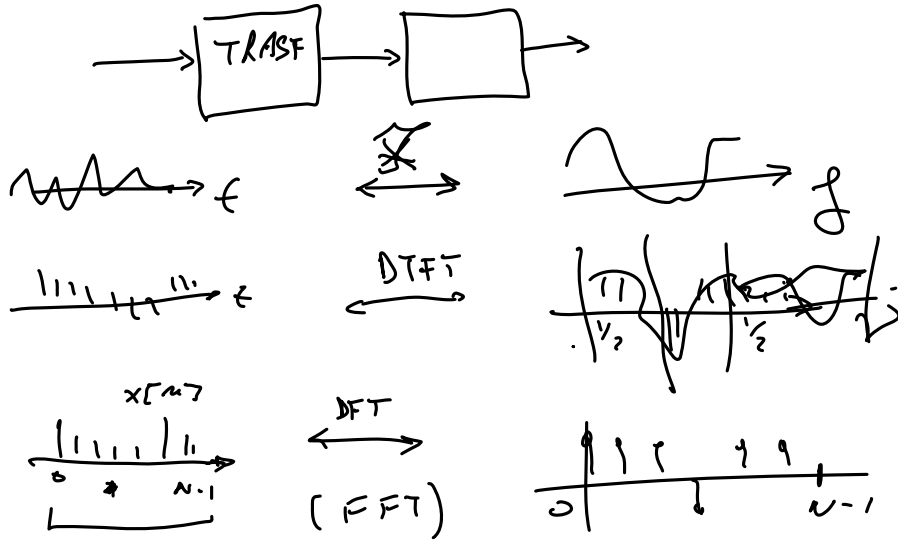
W_N^{kn}

$$\frac{1}{N} \begin{bmatrix} 1 \\ W_N^{-K N} \\ \vdots \\ W_N^{+K N} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$W_N X = X$$

$$\frac{1}{N} W_N^* X = x$$

$\sim \quad \sim$



DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \quad k=0, \dots, N-1$$



IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn} \quad n=0, \dots, N-1$$

DFT

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j \frac{2\pi}{N}} & e^{-j \frac{2\pi}{N} 2} & \dots & e^{-j \frac{2\pi}{N} (N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j \frac{2\pi}{N}} & e^{j \frac{2\pi}{N} 2} & \dots & e^{j \frac{2\pi}{N} (N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

W_N simmetrica

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{j \frac{2\pi}{N}} & e^{j \frac{2\pi}{N} 2} & \dots & e^{j \frac{2\pi}{N} (N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j \frac{2\pi}{N}} & e^{-j \frac{2\pi}{N} 2} & \dots & e^{-j \frac{2\pi}{N} (N-1)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\underline{X} = \underline{W}_N \underline{x}$$

$$\underline{X} = \underline{A} \underline{x}$$

$$\underline{x} = \underline{A}^{-1} \underline{X}$$

$$\underline{x} = \frac{1}{N} \underline{W}_N^* \underline{X}$$

$$\frac{1}{N} \underline{W}_N^* = \underline{W}_N^{-1}$$

Matrice unitaria

$$\underline{A}^{-1} = \underline{A}^H = \underline{A}^{T*}$$

$\underline{X}$ ve Korelezione

$$\underline{X} \rightarrow \underline{W}_N \rightarrow \underline{X}$$

$\underline{x}$ vettore aleatorio

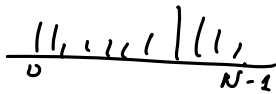
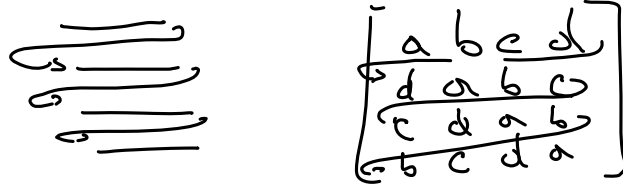


$$E[\underline{x}] = 0 \Rightarrow E[\underline{x}] = E[W_N \underline{x}] = W_N E[\underline{x}] = 0$$

$$R_{\underline{x}} = E[\underline{x} \underline{x}^H] = E[W_N \underline{x} \underline{x}^H W_N^H] = W_N E[\underline{x} \underline{x}^H] W_N^H \\ = W_N R_{\underline{x}} W_N^H \quad \text{diagonale?}$$

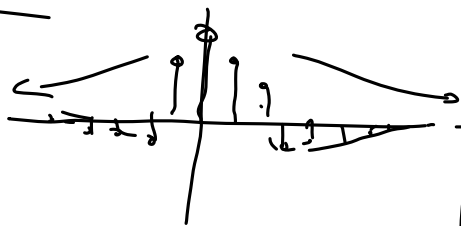
Teorema:

$R_{\underline{x}}$ è diagonale e $R_{\underline{x}}$ è circolante



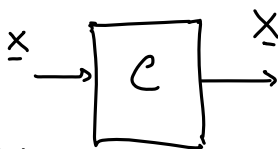
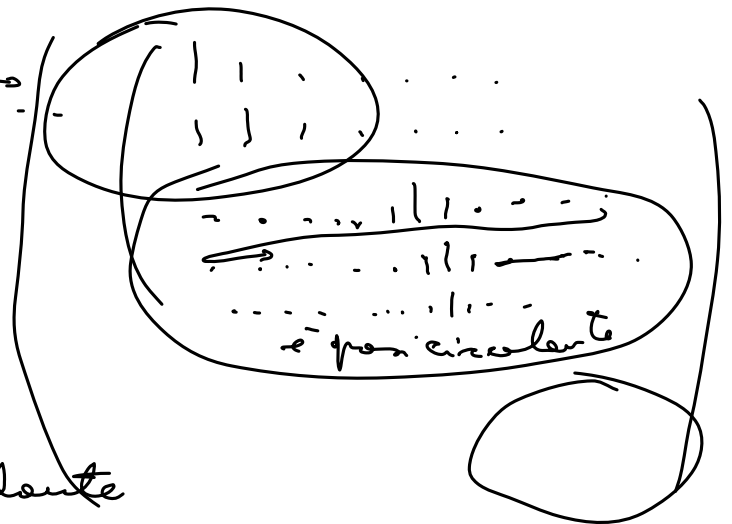
processo staz. SL $R_{\underline{x}} = \begin{bmatrix} r_x(0) & r_x(1) & \dots & r_x(N-1) \\ r_x(-1) & r_x(0) & \dots & r_x(N-2) \\ r_x(-2) & r_x(-1) & r_x(0) & \dots & r_x(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_x(-N+1) & \dots & \dots & \dots & r_x(0) \end{bmatrix}$

$r_x[m] \triangleq E[x[n] x^*[n-m]]$



Quando $R_{\underline{x}}$ per un processo staz. è già circolante

$\Rightarrow$ DFT è decorrelante (quasi)



$\underline{x}$ v.o. vettoriale

$$E[\underline{x}] = 0$$

$$SSL \quad R_{\underline{x}} = E[\underline{x} \underline{x}^H]$$

$$\underline{x} = C \underline{x}$$

Proprietà C $\Rightarrow R_{\underline{x}} = E[\underline{x} \underline{x}^H] = \text{diagonale}$

$$\begin{pmatrix} x_x & 0 \\ 0 & x_x \end{pmatrix}$$

$$C_{cov} = E[(\underline{x} - \mu_x)(\underline{x} - \mu_x)^H] \\ = R_{\underline{x}}$$

$$E[\underline{X} \underline{X}^H] = E[\underline{C} \underline{x} \underline{x}^H \underline{C}^H] = \underline{C} E[\underline{x} \underline{x}^H] \underline{C}^H$$

$$= \underline{C} \underline{R}_x \underline{C}^H = \text{diagonale.}$$

Fattorizzazione spettrale:

$\underline{R}_x$ simmetrica (Hermitiana)

$$\underline{R}_x \geq 0$$

$\underline{R}_x = \underline{F} \underline{F}^H$ Hottelato: $\underline{F}$ ortogonale

$$\underline{C} \underline{F} \underline{F}^H \underline{C}^H = \underline{D}$$

$$\underline{C} = \underline{F}^{-1} \Rightarrow \underline{F}^{-1} \underline{F} \underline{F}^H \underline{F}^{-1} = \underline{I}$$

$$\underline{C} = \alpha \underline{F}^{-1} \Rightarrow \alpha \underline{F}^{-1} \underline{F} \underline{F}^H \underline{F}^{-1} \alpha = \alpha^2 \underline{I}$$

caso reale

$$\underline{R}_x = \underline{L} \underline{\Lambda} \underline{L}^T$$

F.N. di Cholesky

$$\underline{F} = \underline{L}$$

$$\underline{F}^{-1} = \underline{L}^{-1} \text{ (triangolare)}$$

$$\begin{pmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{C} = \underline{L}^{-1} \Rightarrow \underline{C} = \underline{\Lambda}^{-1/2} \underline{\Lambda}^{1/2} \underline{L}^{-1}$$

$$\underline{R}_x = \underline{L} \underline{\Lambda} \underline{L}^T = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \begin{pmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{F} = \underline{L} \underline{\Lambda}^{1/2} \Rightarrow \underline{C} = \underline{\Lambda}^{-1/2} \underline{L}^{-1}$$

$$\underline{R}_x = E[\underline{X} \underline{X}^T] = \underline{C} \underline{R}_x \underline{C}^T = \underline{\Lambda}^{-1/2} \underline{L}^{-1} \underline{L} \underline{\Lambda} \underline{L}^T \underline{\Lambda}^{-1/2} = \underline{I}$$

$$\underline{F} = \underline{L} \Rightarrow \underline{C} = \underline{L}^{-1}$$

$$\underline{R}_x = E[\underline{X} \underline{X}^T] = \underline{I}$$

FATTORIZZAZIONE SPECTRALE

$$\underline{R}_x = \underline{Q} \underline{\Lambda} \underline{Q}^H$$

$\underline{Q}$ ortogonale

$$R_x = Q \Lambda Q^H$$

autovetori di R_x $\uparrow$ autovetori

$$[q_0 q_1 \dots q_{N-1}] \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{pmatrix} \begin{bmatrix} q_0^H \\ q_1^H \\ \vdots \\ q_{N-1}^H \end{bmatrix}$$

Q ortogonale

$$Q^H Q = I$$

$$Q^{-1} = Q^H$$

① $F = Q$ $C = Q^{-1} = Q^H$!!

$\underline{x} \rightarrow \boxed{Q^H} \rightarrow \underline{x}$ $E[\underline{x} \underline{x}^H] = C R_x C^H = \cancel{Q^H Q} \Lambda \cancel{Q Q^H} = \Lambda$

② $F = Q \Lambda^{1/2}$ $C = \Lambda^{-1/2} Q^H$

$\underline{x} \rightarrow \boxed{\Lambda^{1/2} Q^H} \rightarrow \underline{x}$ $E[\underline{x} \underline{x}^H] = \Lambda^{-1/2} \cancel{Q^H Q} \Lambda \cancel{Q Q^H} \Lambda^{1/2} = I$

$\underline{x} \rightarrow \boxed{Q^H} \rightarrow \underline{x} = Q^H \underline{x}$

$\int |x(t)|^2 dt = \|x\|^2$
 $\xrightarrow{\varepsilon_x} \xrightarrow{f}$

$\underline{\varepsilon}_x = E[\underline{x}^H \underline{x}] = E[\underline{x}^H \underbrace{Q Q^H}_I \underline{x}] = E[\underline{x}^H \underline{x}] = \underline{\varepsilon}_x$

$E[\underline{x} \underline{x}^H] = \Lambda = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{pmatrix} = \sum_{i=0}^{N-1} \lambda_i$

$R_x = Q \Lambda Q^H$

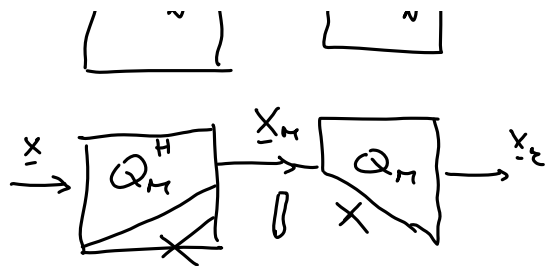


$\underline{x} \rightarrow \boxed{Q^H} \rightarrow \underline{x} \rightarrow \boxed{Q} \rightarrow \underline{x}$

Ente ricorrenza

$\underline{x} \rightarrow \boxed{Q_N^H} \rightarrow \underline{x} \rightarrow \boxed{Q_N} \rightarrow \underline{x}$

KL T (discrete)
Karhunen-Loève TRANSFORM



Karhunen-Loève TRANSFORM
 conservo $\lambda_0, \lambda_1, \dots, \lambda_{N-1}$
 resto $\lambda_N, \dots, \lambda_{N-1}$

$$R_x = Q_N \Lambda_N Q_N^H = [Q_0 \dots Q_{N-1}] \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{N-1} \end{bmatrix} \begin{bmatrix} Q_0^H \\ \vdots \\ Q_{N-1}^H \end{bmatrix}$$

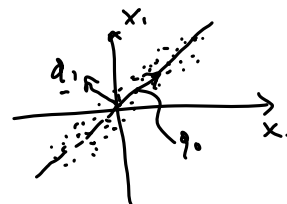
$\underbrace{\begin{matrix} | & | & | & | \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{matrix}}_{\text{diagonal}} \equiv \text{diagonal matrix}$

$x_{\hat{}} = Q_N Q_N^H x$ $E[\|x_{\hat{}} - x\|^2]$? Error quadratico medio della ricostruzione

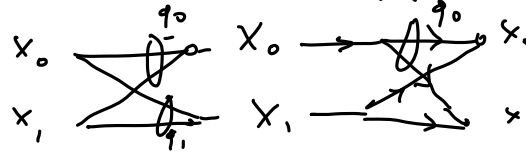
$E[(x_{\hat{}} - x)^H (x_{\hat{}} - x)] = 0 \dots 0 = \sum_{i=N}^{N-1} \lambda_i$

PCA Principal Component Analysis

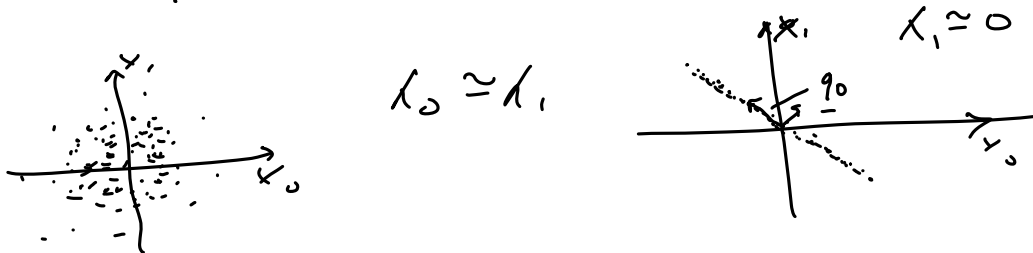
Esempio x $N=2$ $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ $E[x] = 0$



$R_x = [q_0 \ q_1] \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} q_0^T \\ q_1^T \end{bmatrix}$



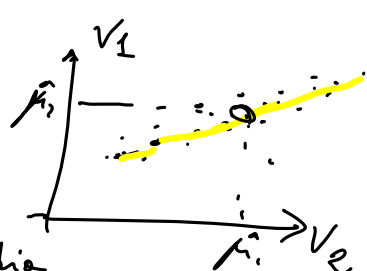
Q^T

$$X = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} q_0^T x \\ q_1^T x \end{bmatrix}$$


Lineare pratica

⇓

rimuovo la media



Le punti:

$$\begin{bmatrix} v_1[1] & \dots & v_1[L] \\ v_2[1] & \dots & v_2[L] \end{bmatrix}$$

$$\hat{\mu}_1 = \frac{1}{L} \sum_{i=1}^L v_1[i]$$

$$\hat{\mu}_2 = \frac{1}{L} \sum_{i=1}^L v_2[i]$$

$$y_1 = v_1 - \mu_1$$

$$y_2 = v_2 - \mu_2$$

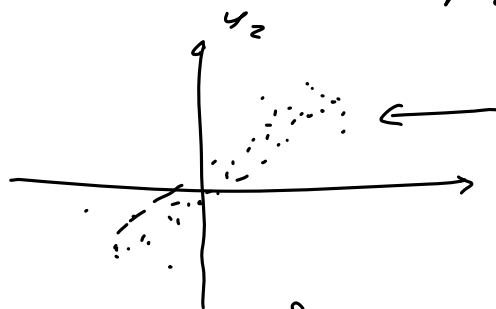
$$\hat{R}_x = \frac{1}{L} \sum_{i=1}^L \underline{y}[i] \underline{y}[i]^T$$

$$\hat{R}_x = \underline{Q} \underline{\Lambda} \underline{Q}^T$$

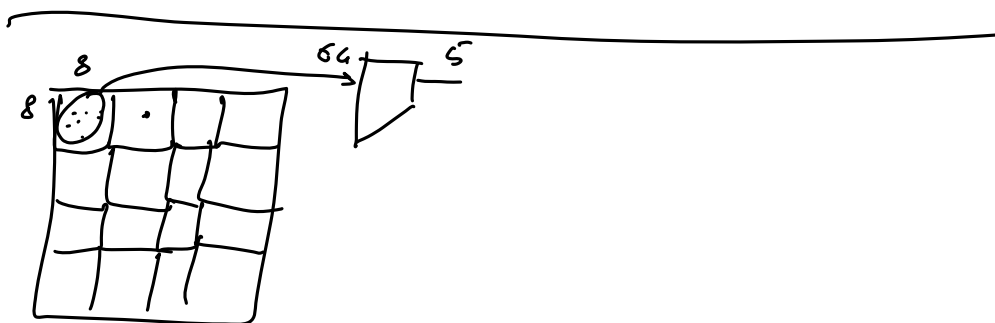
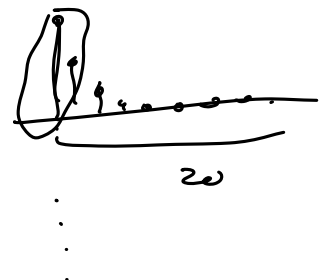
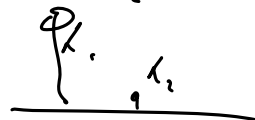
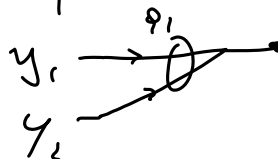
$$[1] [1] [0] [0]$$

$$[q_1, q_2] [\lambda_1, \lambda_2] [q_1^T, q_2^T]$$

$$\mu_2 = \frac{1}{L} \sum_{i=1}^L v_2[i]$$



$$\begin{bmatrix} y_1[1] & \dots & y_1[L] \\ y_2[1] & \dots & y_2[L] \end{bmatrix}$$



Trasf. di Fourier (Proprietà)

STFT, DFT

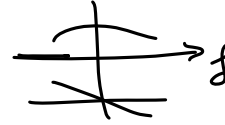
Trasf. Discrete: Decrelante

$R_x \rightarrow f$ D. spettrale



Altre trasformate

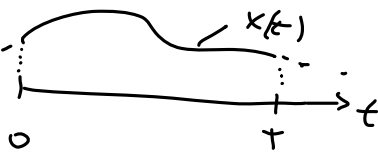
$$\underset{\text{reale}}{x(t)} \longleftrightarrow \underset{\substack{\text{complessa} \\ \text{(Hermitiana)}}}{X(f)}$$



Trasformata di Hartley

$$X_H(f) = \int_{-\infty}^{\infty} x(t) (\underbrace{\cos(2\pi f t) + j \sin(2\pi f t)}_{\text{cos.}}) dt$$

SHORT-TIME TRANSFORM



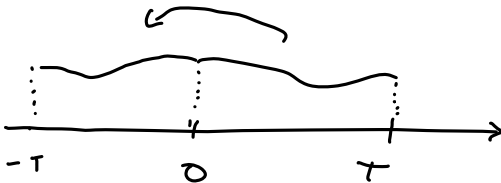
$$X(f) = \int_0^T x_T(t) e^{-j2\pi f t} dt$$

Short-time Fourier Transform.

Temp.-reverse $x_T(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} df$

TRASFORMATA COSENO

JPEG (Trasf. coseno)

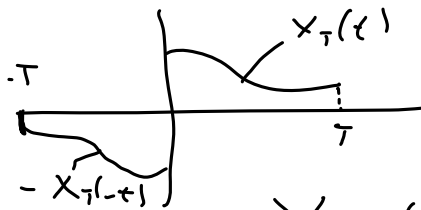


$x(t)$ reale $\rightarrow X(f)$ Hermitica.
Perciò $\rightarrow X(f)$ reale.

$$X_{\cos}(f) = \int_0^T x(t) \cos 2\pi f t dt \quad f \in \mathbb{R}^+$$

$$x(t) = 4 \int_0^{\infty} X_{\cos}(f) \cos 2\pi f t df \quad t \in [0, T]$$

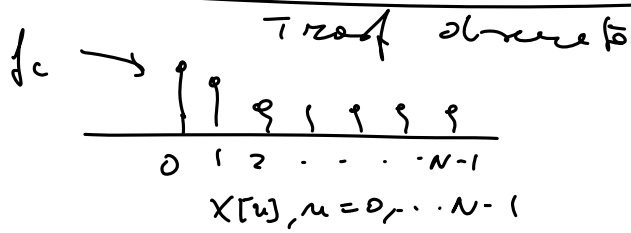
TRASFORMATA SENO



$\rightarrow \mathcal{F} \rightarrow$

$$X_{\sin}(f) = - \int_0^T x(t) \sin 2\pi f t dt$$

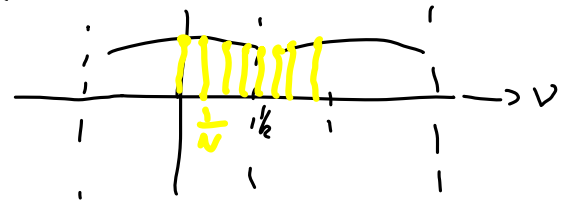
$f_c \rightarrow f_0$ Trasf. discreta AFT



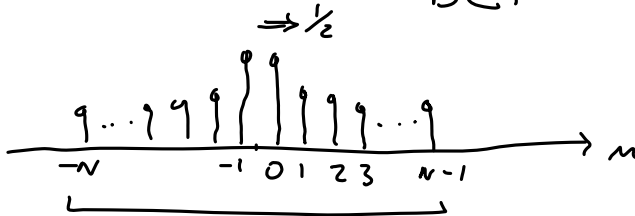
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}$$

$k=0, \dots, N-1$



TRASFORMATA COSENO (Discrete)
DCT



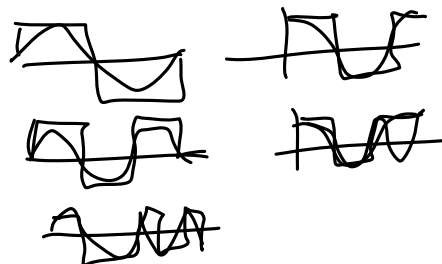
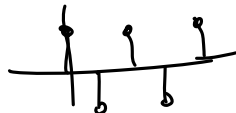
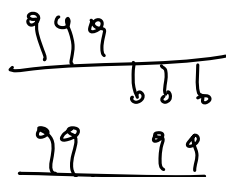
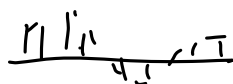
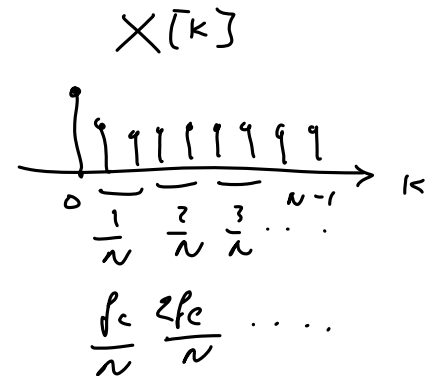
$$V[k] = e^{-j \frac{2\pi}{2N} k \frac{1}{2}} \text{DCT}(\cdot) = e^{-j \frac{\pi}{2N} k} \sum_{n=-N}^{N-1} v[n]$$

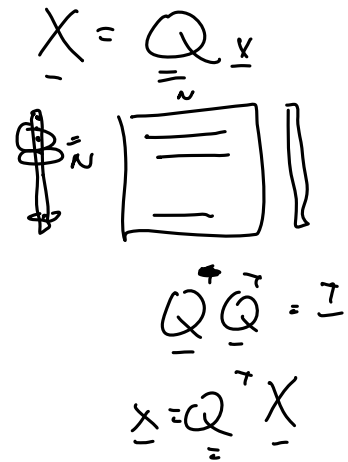
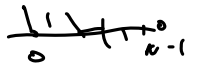
$\dots =$

$$\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \quad \underline{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} Q_c \end{bmatrix} \underline{x}$$

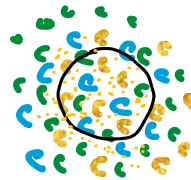
$$\underline{x} \begin{bmatrix} Q_c \end{bmatrix} \rightarrow \underline{X} \begin{bmatrix} Q_c^T \end{bmatrix} \rightarrow \underline{x} \quad Q_c Q_c^T = I \quad \underline{x} = Q_c^T \underline{X}$$





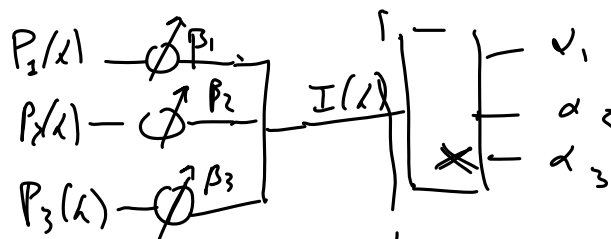
$$\underline{R}_x = E[\underline{x} \underline{x}^T] \rightarrow \underline{R}_x : Q \Lambda Q^T$$

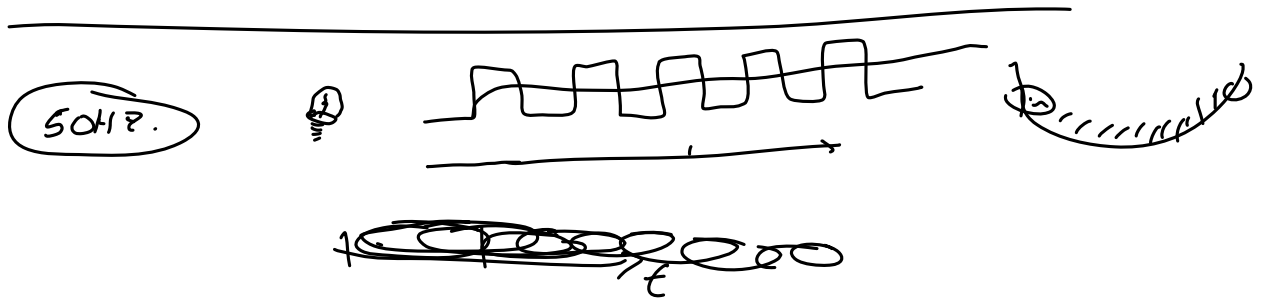
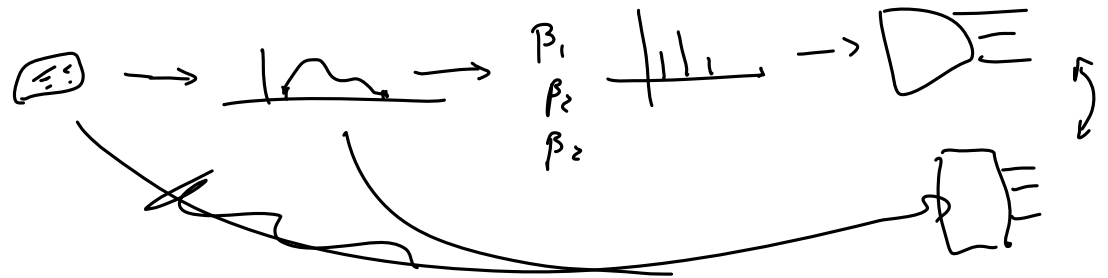
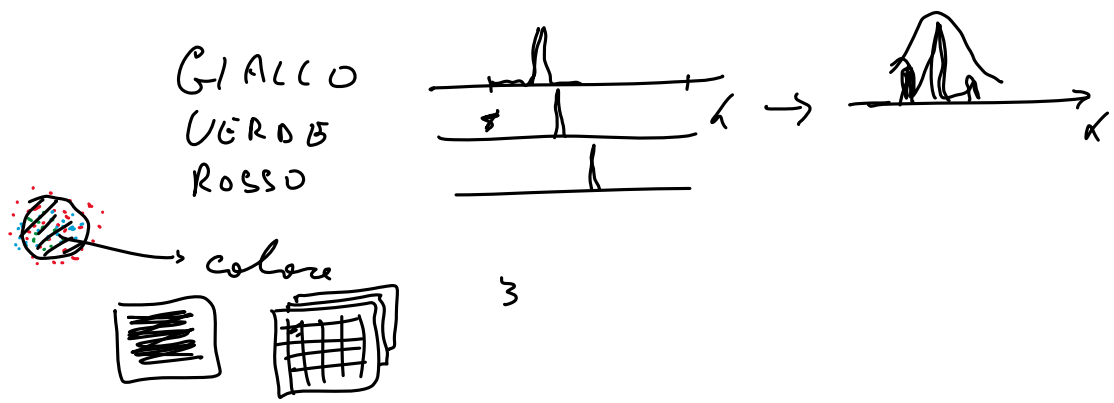
$$\underline{f(x, y, t)}$$

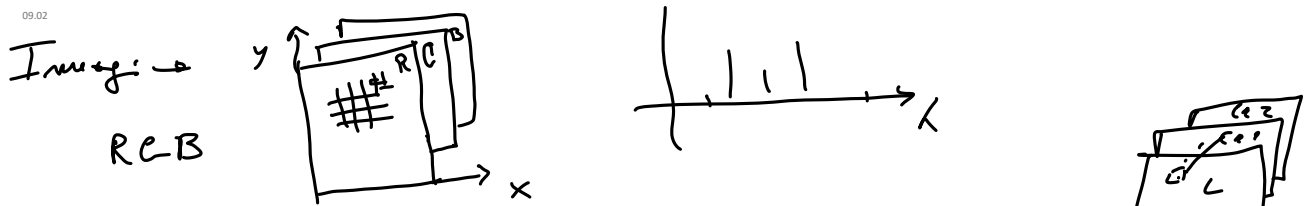


A diagram illustrating a signal processing step. On the left, a plot shows a noisy signal $x(t)$ over time t . An arrow points from this plot to a rectangular block. Three lines emerge from the right side of the block, labeled α_1 , α_2 , and α_3 . A final arrow points to the right, indicating the next step in the process.

REB







Luminanza
Crominanza 1
Crominanza 2

$$L = \alpha_1 R + \alpha_2 G + \alpha_3 B$$

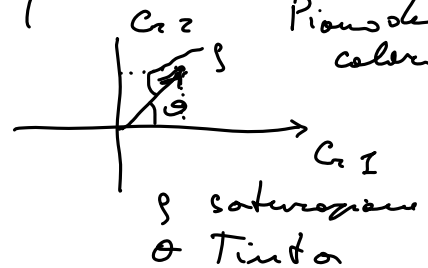
$$C_1 = \beta_1 R + \beta_2 G + \beta_3 B$$

$$C_2 = \gamma_1 R + \gamma_2 G + \gamma_3 B$$

$\rightarrow$ B/W

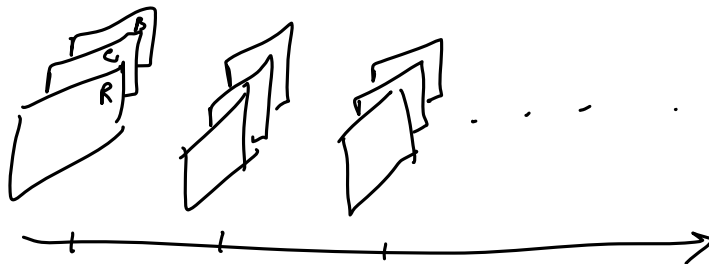
colore.

Più o meno
colore.



$$\begin{bmatrix} L \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

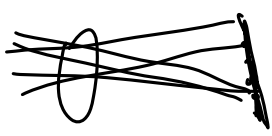
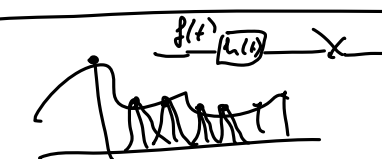
$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} L \\ C_1 \\ C_2 \end{bmatrix}$$



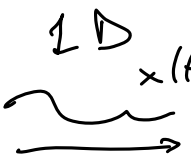
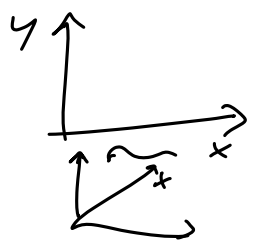
$\frac{1}{T}$ intervalli
di tempo

$\frac{1}{T}$ frequenza di frame
frame rate

30 immagini/sec.
 $T = \frac{1}{30}$ sec.



$f(x,y)$



$$f_c = \frac{1}{T} > 2 f_{max}$$

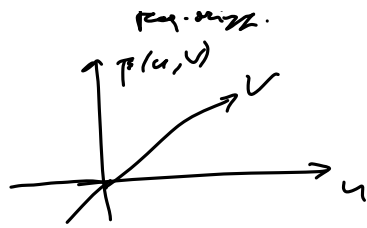


$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} e^{-j2\pi vy} dx dy$$

$\uparrow$ freq. orizz.
 $\uparrow$ freq. vert. $\uparrow$ u, v

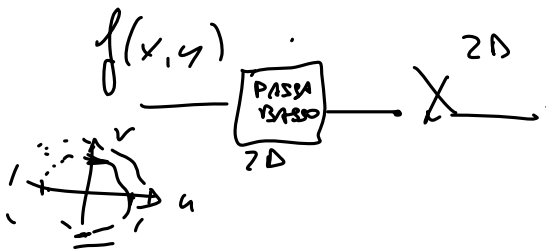
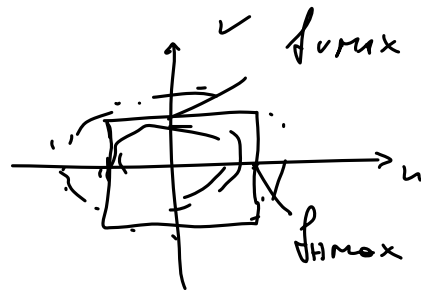
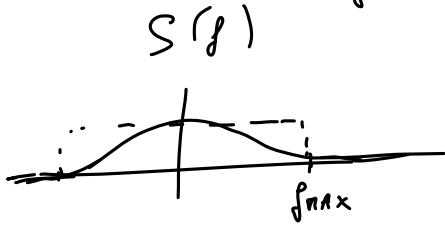
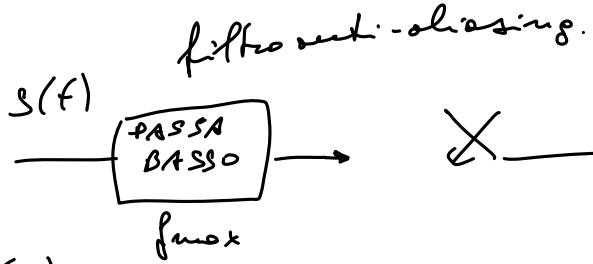
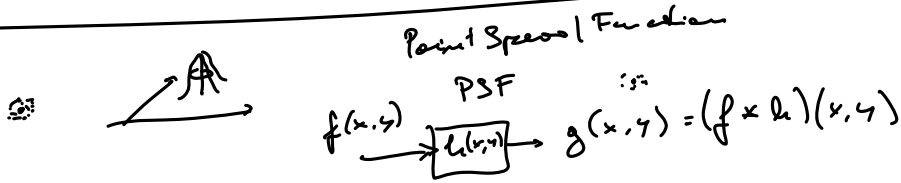
TRASF. DI
FOURIER 2D

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi ux} e^{j2\pi vy} du dv$$



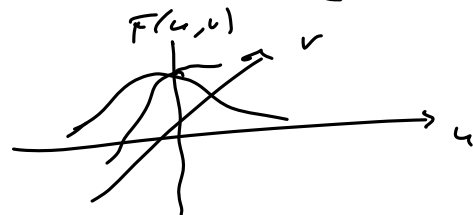
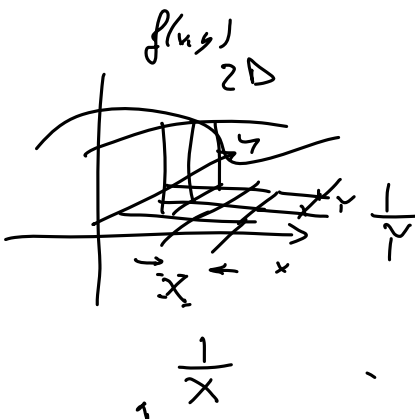
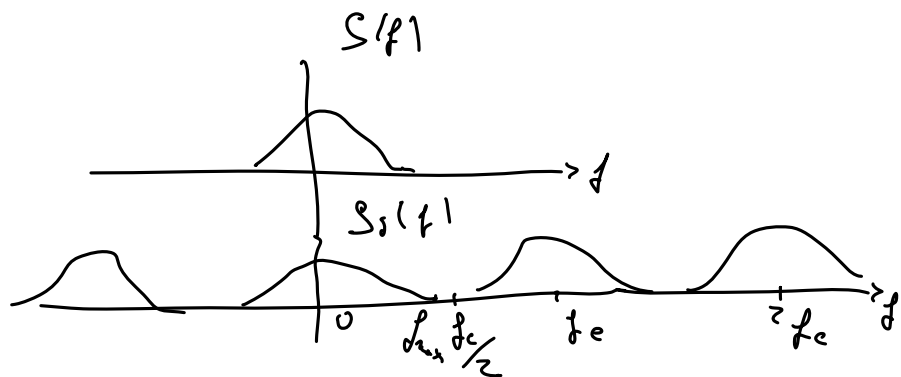
FOURIER TRANSFORM

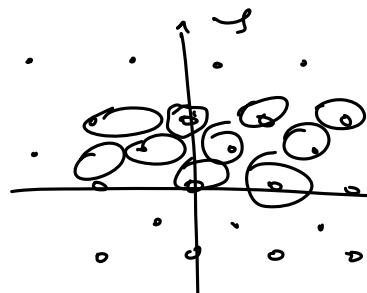
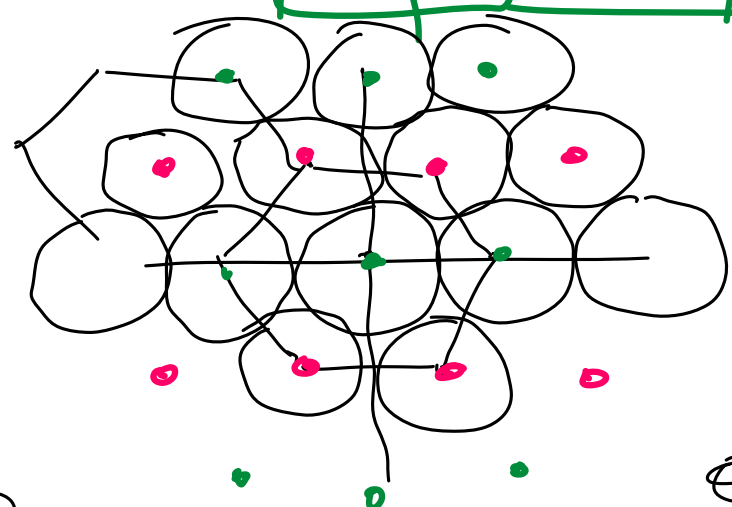
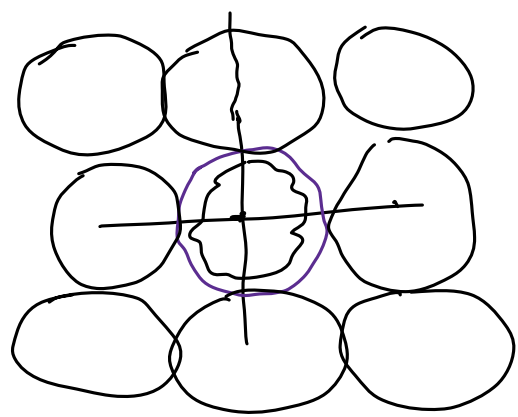
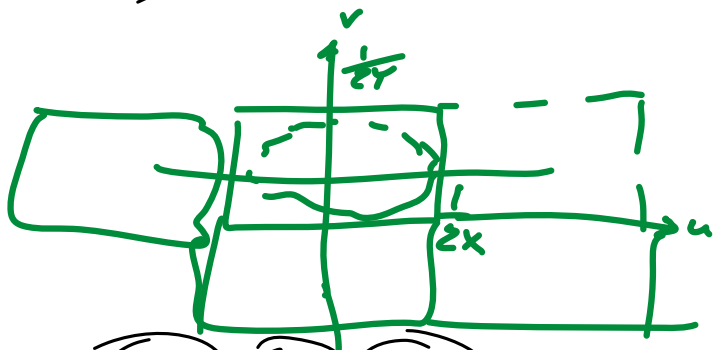
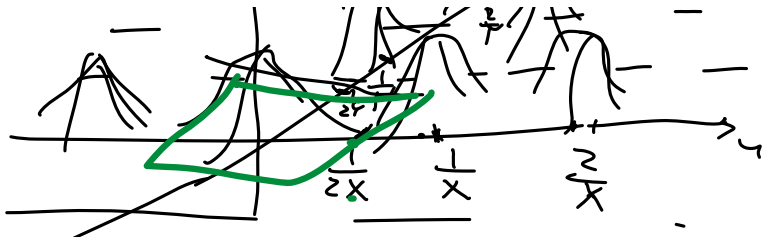
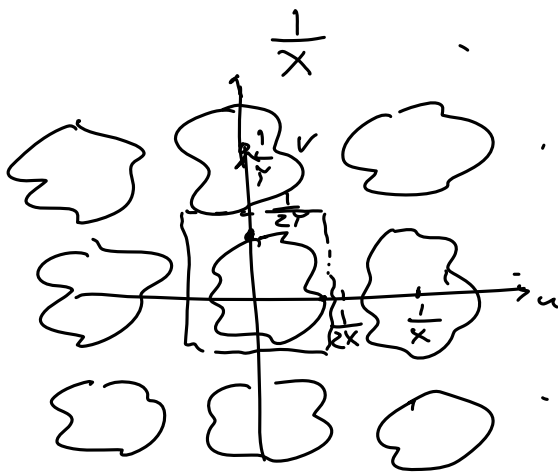
$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv$$



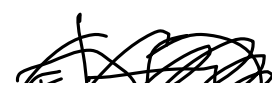
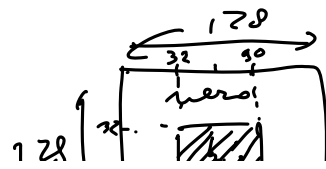
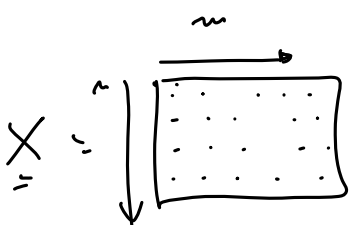
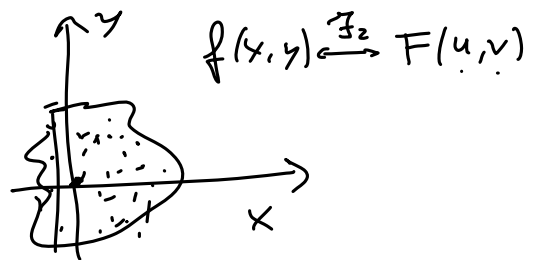
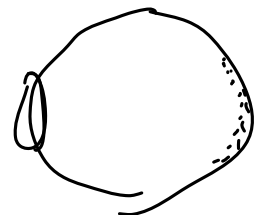
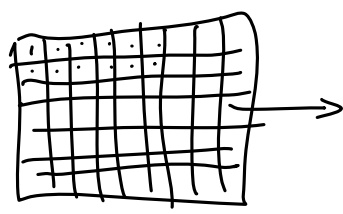
$$\frac{1}{X} \geq 2f_{max} \quad \frac{1}{Y} \geq 2f_{max}$$

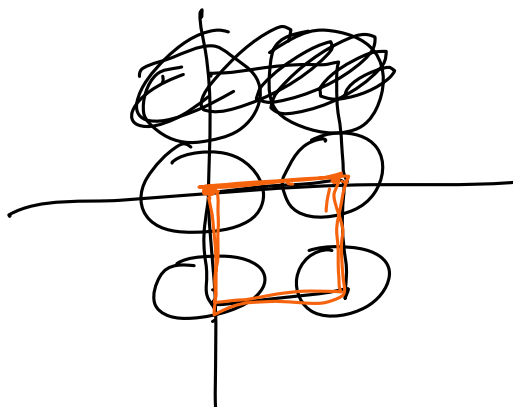
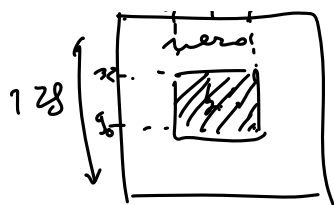
1 D

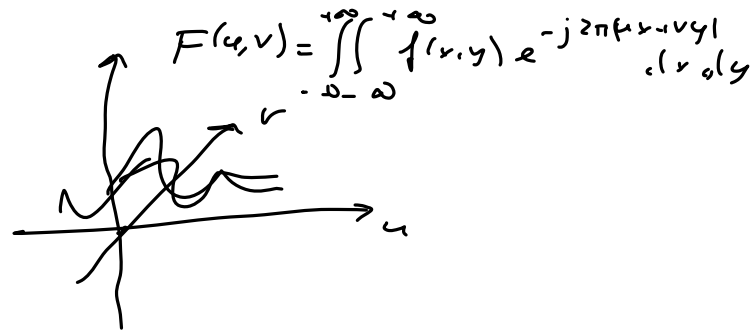
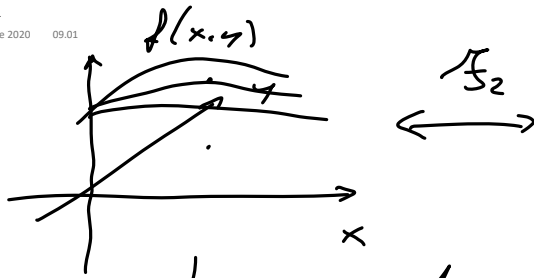




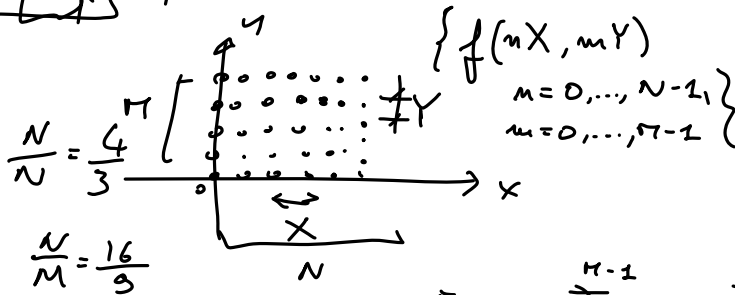
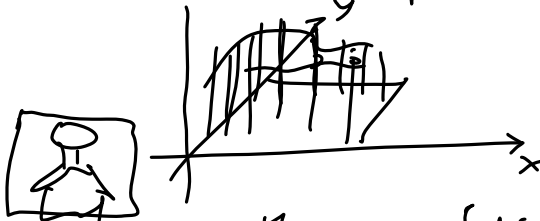
ESACRONALC





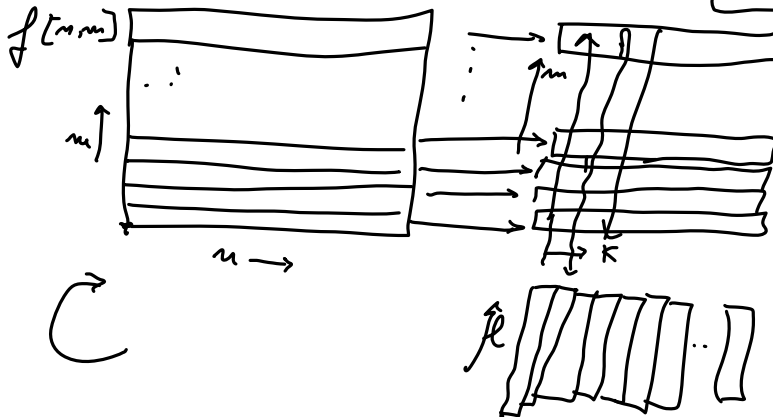


↓ componenti
e frequenze.

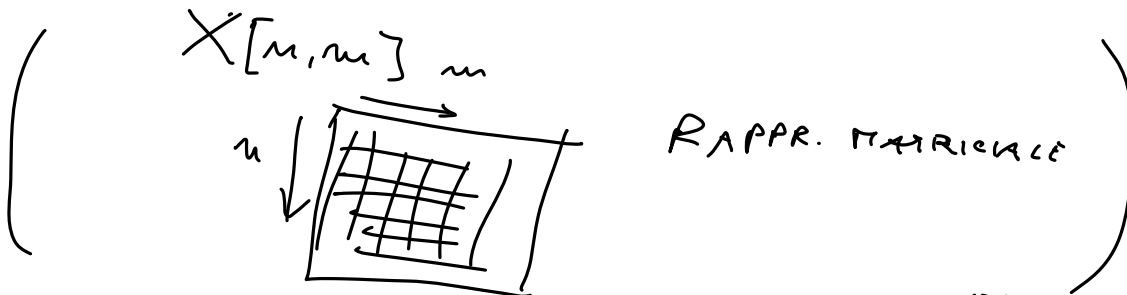


$$DFT_2 \quad F[k, l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[n, m] e^{-j2\pi(\frac{nk}{N} + \frac{ml}{M})}$$

$$F[k, l] = \sum_{m=0}^{M-1} e^{-j2\pi \frac{ml}{M}} \underbrace{\sum_{n=0}^{N-1} f[n, m] e^{-j2\pi \frac{nk}{N}}}_{DFT_1}$$



DFT (FFT)



RAFFA. MATRICE

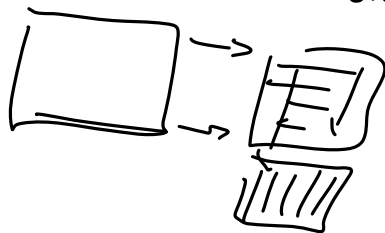
$$F[k, l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[n, m] \underbrace{A[n, k] B[m, l]}_{\text{Kernel separabile.}}$$

$$= \sum_{m=0}^{M-1} B[m, l] \sum_{n=0}^{N-1} f[n, m] A[n, k]$$

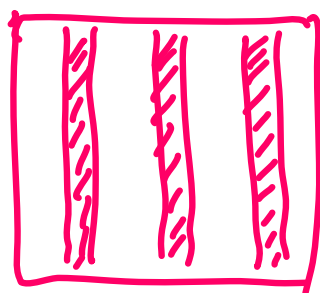
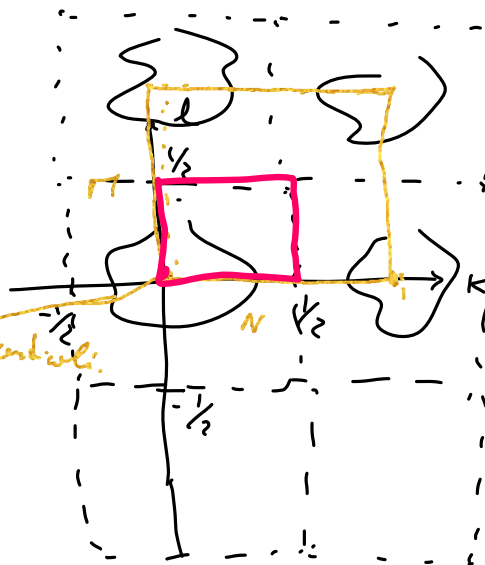
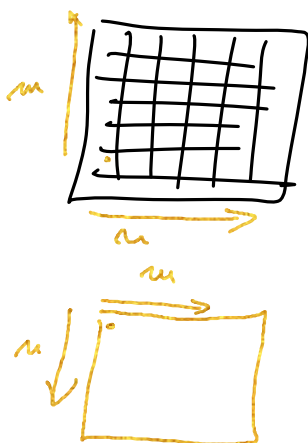
$$= \sum_{m=0}^{M-1} B[m, l] \underbrace{\sum_{n=0}^{N-1} f[m, n] A[n, k]}_{\text{Test per righe.}}$$

column x column.

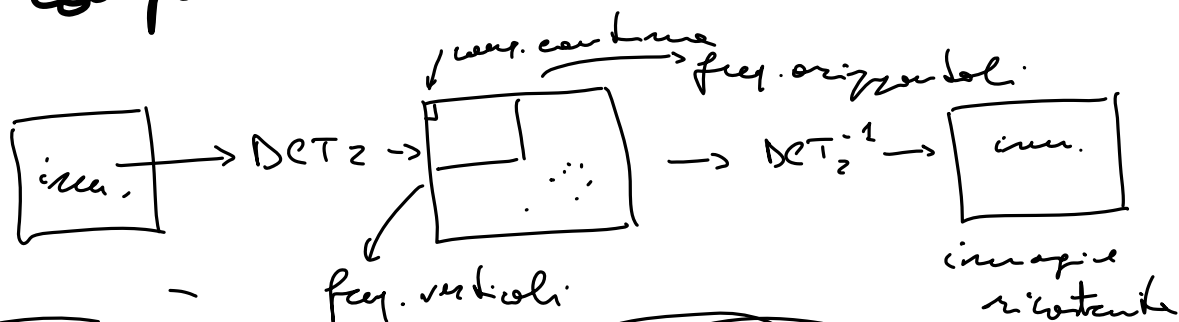
$A = B$



$A \text{ DET}$



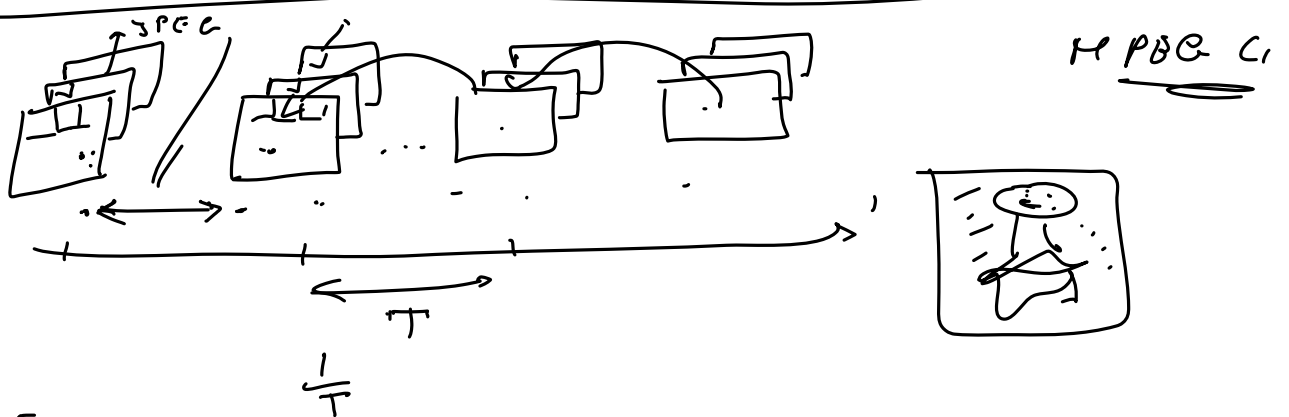
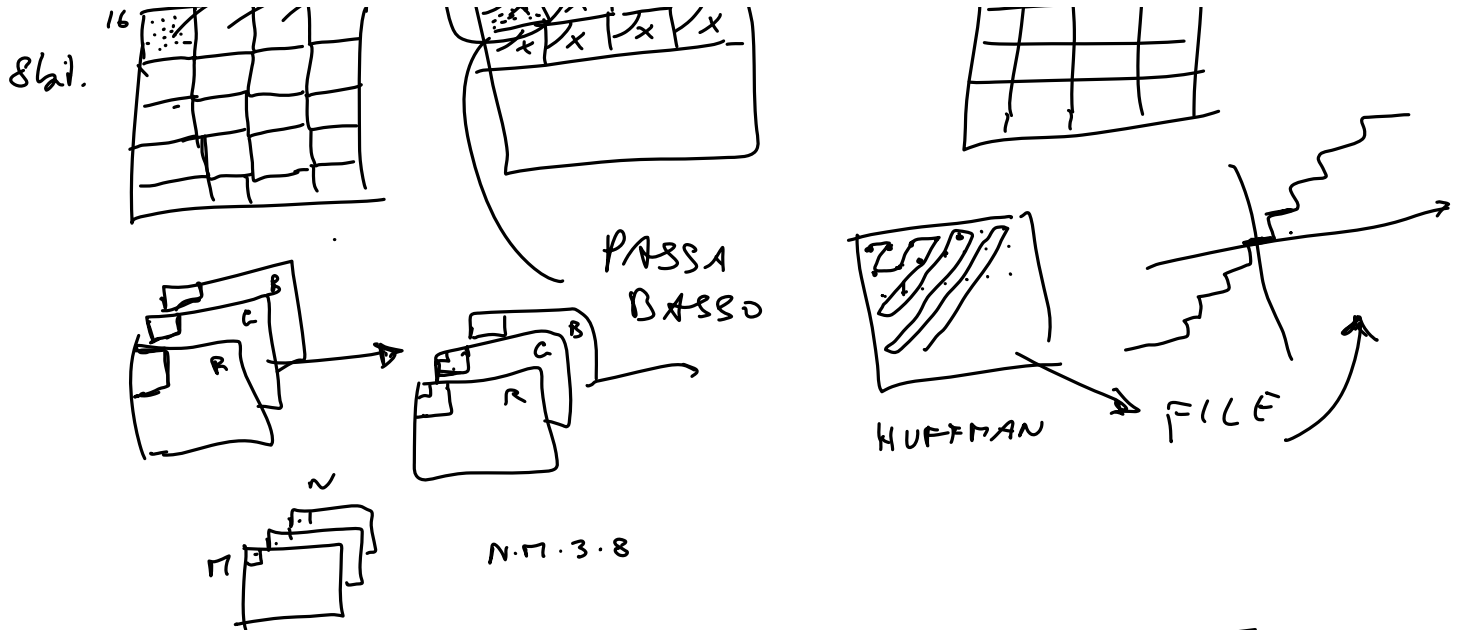
Compression -



JPEG

8bit.



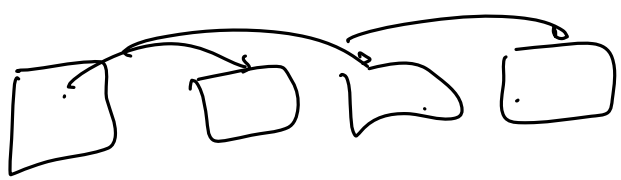


500
400

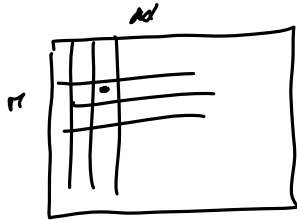
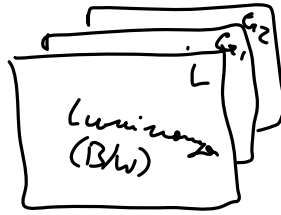
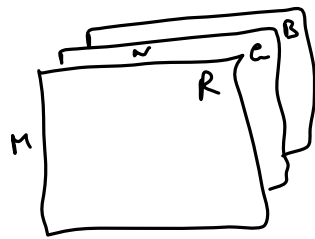
1/50

2-3
400.500.8.3.50 lat/sec.
240 MHz/sec.

COP

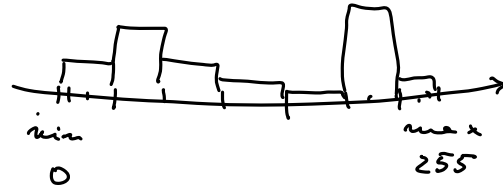


Elab. immagini



$f(x, y)$

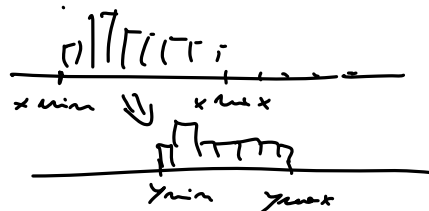
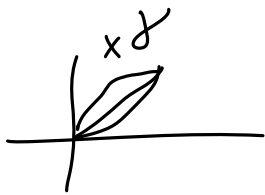
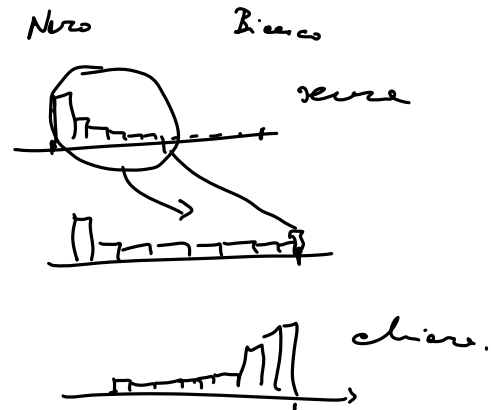
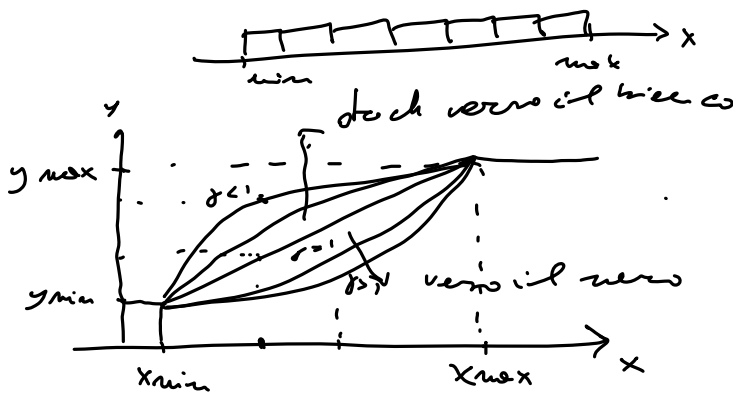
(8 bit)
 f Quantizzato su un range finito



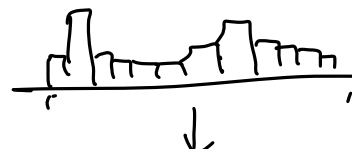
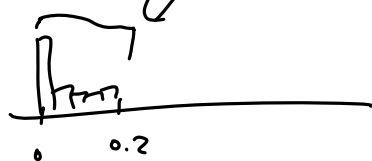
ISTOGRAMMA

→ CONTRASTO

MASSIMO CONTRASTO : istogramma piatto



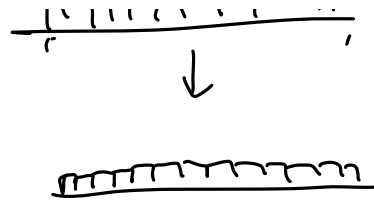
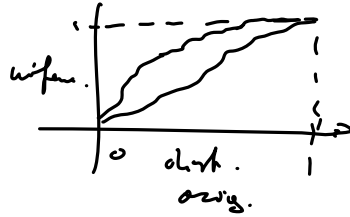
$$y = y_{min} + (y_{max} - y_{min}) \left[\frac{x - x_{min}}{x_{max} - x_{min}} \right]^\gamma$$



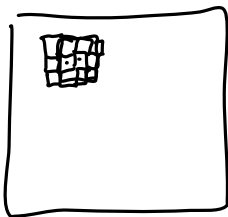
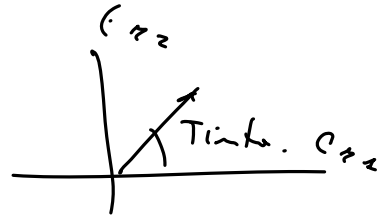
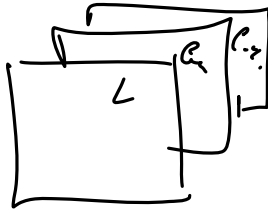
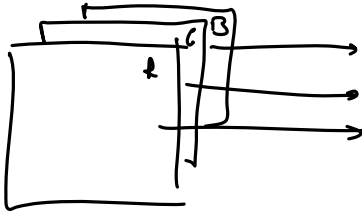
pdf



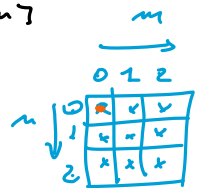
EQUALIZZAZIONE



EQUALIZED.
CELL
ISTOGRAMMA



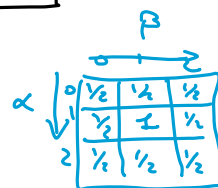
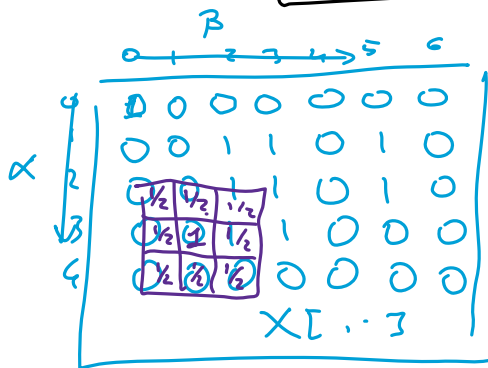
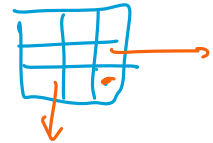
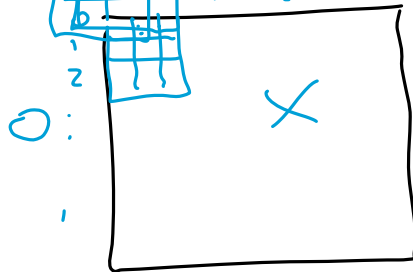
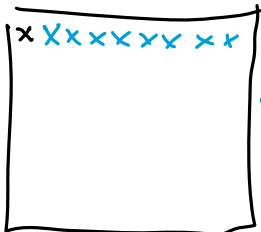
CONVOLUTIONE
LINEARE 2D



$$y[n,m] = \sum_{\beta} \sum_{\alpha} h[\alpha, \beta] x[n-\alpha, m-\beta]$$

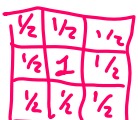
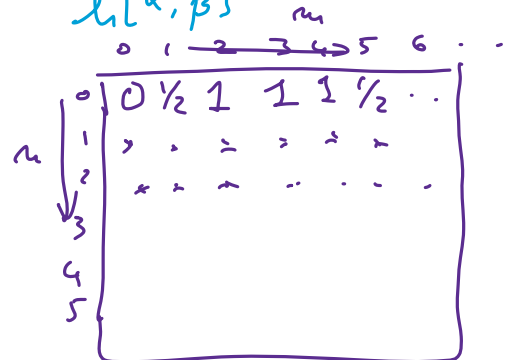


$$= \sum_{\beta} \sum_{\alpha} x[\alpha, \beta] h[n-\alpha, m-\beta]$$

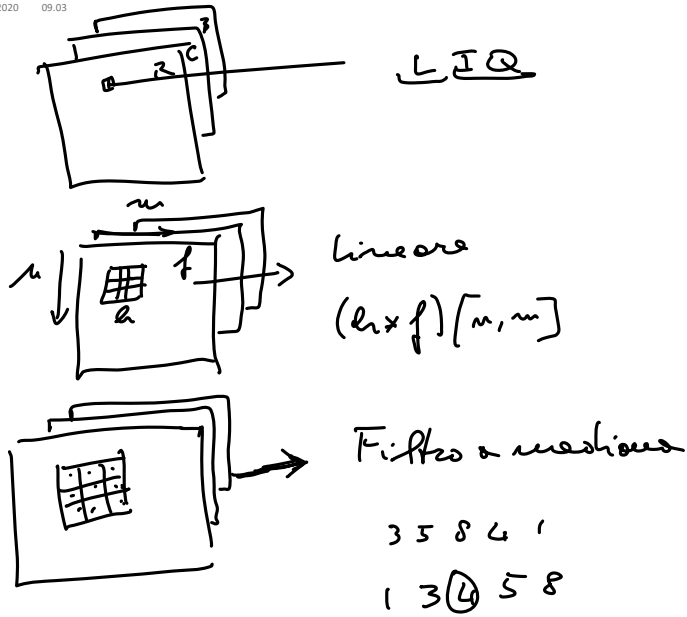


$h[n-\alpha, m-\beta]$

$h[\alpha, \beta]$

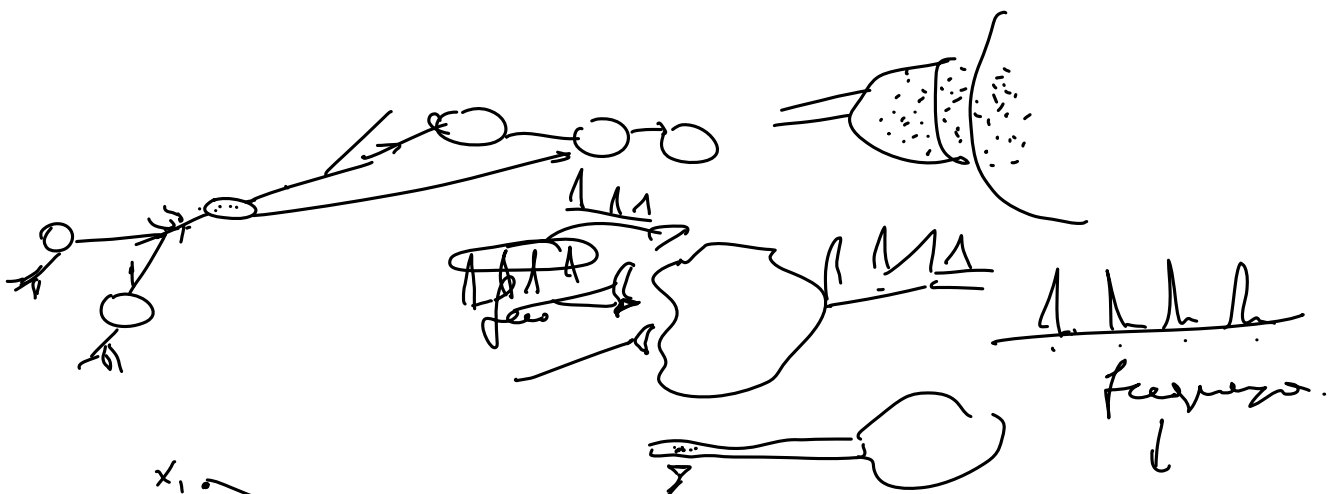
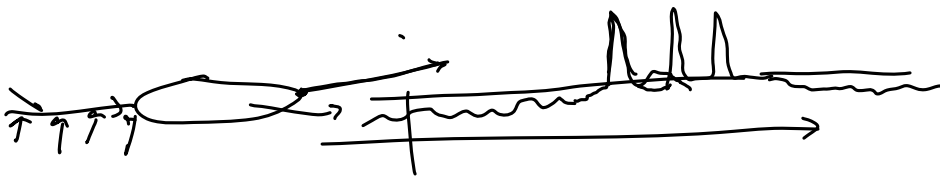
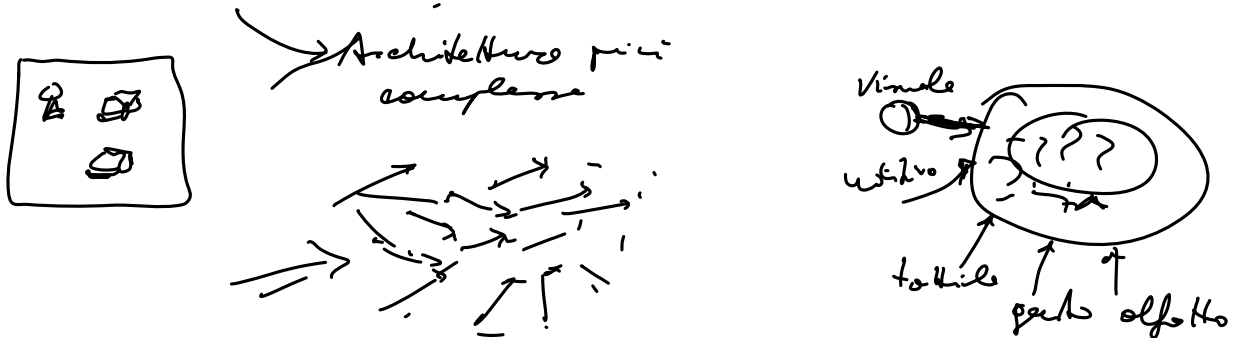


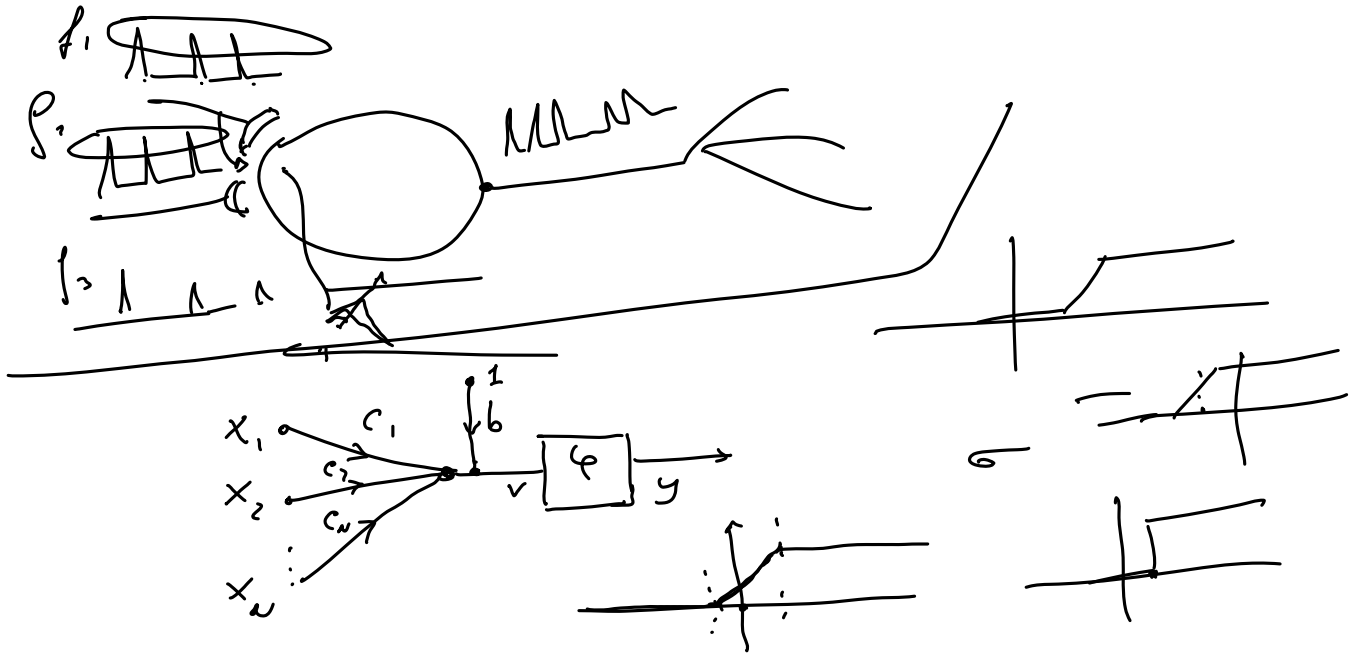
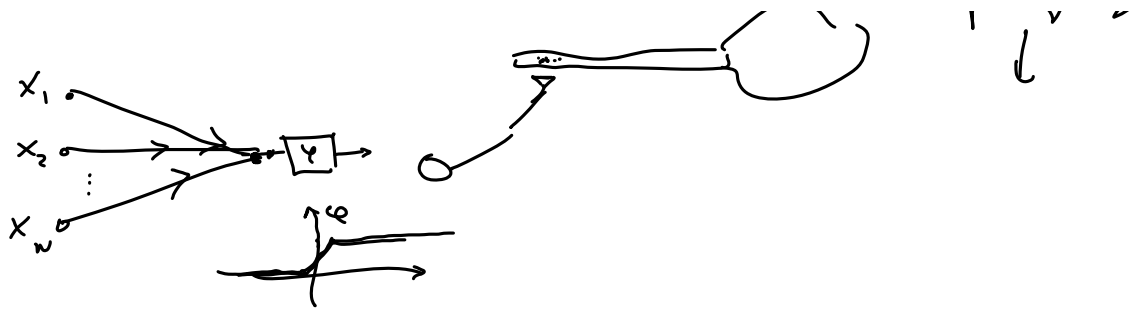
$$\text{DTFT}_{2D}(y[n,m]) = \text{DTFT}_2(h[n,m]) \cdot \text{DTFT}(x[n,m])$$



Reti neurali / Machine Learning

Sistemi di ENS : { operazioni lineari
 non lineari





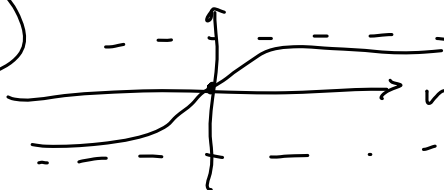
$$v = c_1 x_1 + c_2 x_2 + \dots + c_n x_n + b$$

$$y = \varphi(v)$$

$$\varphi(v) = \frac{1}{1 + e^{-av}}$$

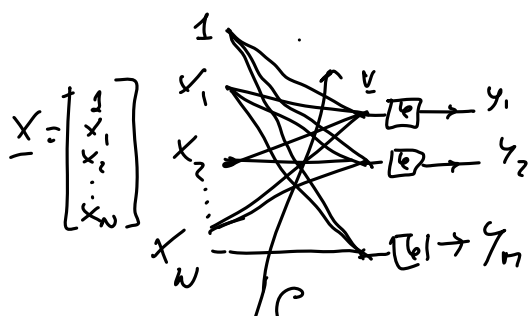


$$\varphi(v) \approx \frac{1}{1 + e^{-av}}$$



$$\ln\left(\frac{1}{1 + e^{-av}} - \frac{1}{2}\right) = \ln\left(\frac{2 - 1 - e^{-av}}{2(1 + e^{-av})}\right) = \ln\left(\frac{1 - e^{-av}}{2(1 + e^{-av})}\right) = \ln\left(\frac{e^{-\frac{av}{2}}(e^{\frac{av}{2}} - e^{-\frac{av}{2}})}{2e^{-\frac{av}{2}}(e^{\frac{av}{2}} + e^{-\frac{av}{2}})}\right)$$

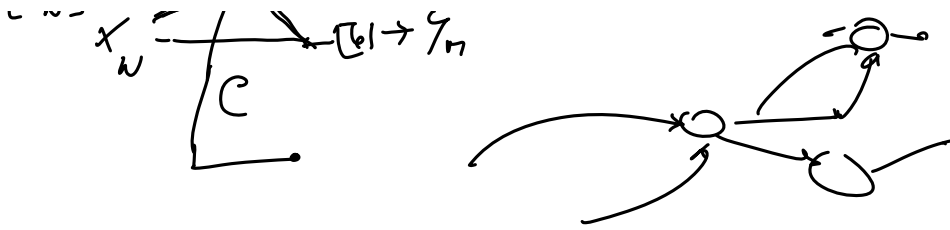
$$2(\varphi(v) - \frac{1}{2}) = \sqrt{\ln \frac{av}{2}}$$



$$\underline{v} = \underline{C} \underline{x}$$

$$\underline{y} = \varphi(\underline{v})$$

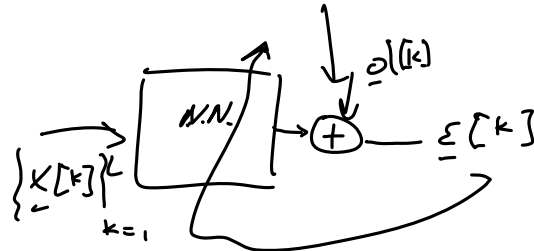




A DATTIVITA'

LEARNING APPRENDIMENTO

UN INSEGNANTE (TEACHER)



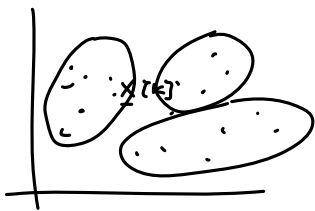
APPRENDIMENTO SUPERVISIONATO

$\{x[k], d[k] \mid k=1, \dots, L\}$
TRAINING SET

$\{x[k] \mid k=L+1, \dots\}$
TEST SET CON

UNSUPERVISED (SENZA INSEGNANTE)
NO TEACHER

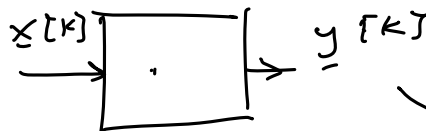
$y \in \{1, 2, 3\}$



$\{x[k] \mid k=1, \dots, L\}$

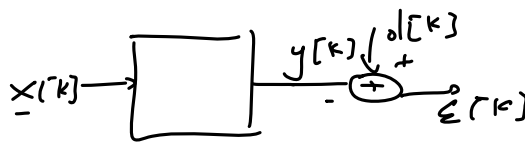
non c'è il
desiderato.

$\{x[k] \mid k=L+1, \dots\}$
TEST SET



frequenza di
costo nella
rappresentazione.

GENERALIZZAZIONE



$d[k] \in \begin{cases} \text{insieme discreto} \\ \text{insieme continuo} \end{cases}$

$d[k] \in \{a_1, a_2, \dots, a_m\}$ discreti

PATTERN CLASSIFICATION
CLASSIFICAZIONE

$x[k] \rightarrow d[k]$

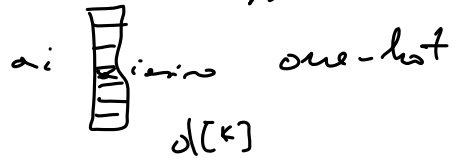
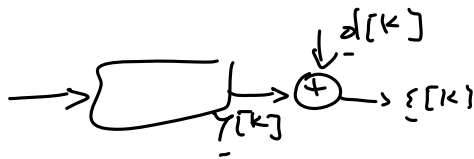


persona $a_1, 1$
libro $a_2, 2$
risultato $a_3, 3$
...



$x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix}$

stiche tra $\rightarrow$ valore numerico
o una rappresentazione

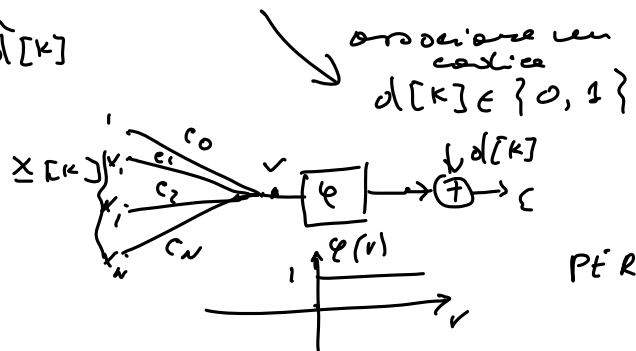
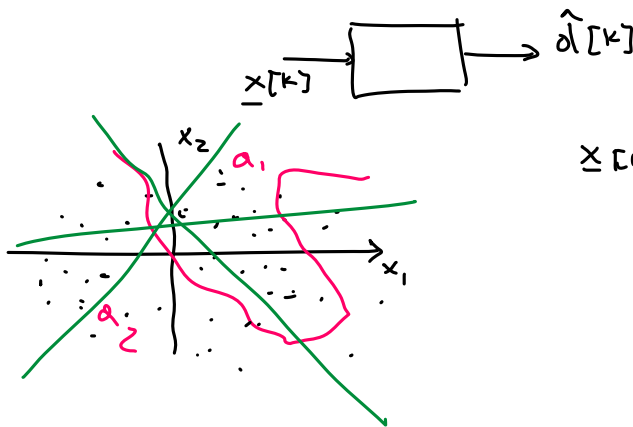


$d[k]$ è vettore costante

REGRESSIONE

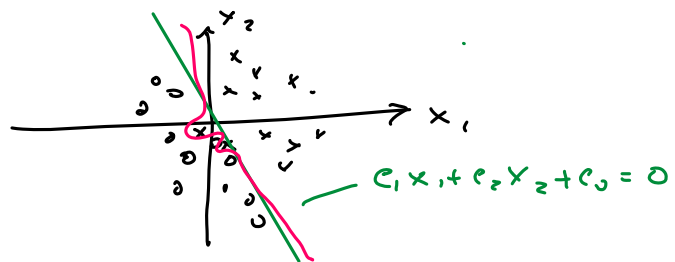
Esempio PATTERN CLASSIFICATION (BINARIO)

$$\underline{x}[k] \in \mathcal{X} \subseteq \mathbb{R}^N \quad d[k] \in \{a_1, a_2\}$$



PERCETTRONE

$$v = c_1 x_1 + c_2 x_2 + \dots + c_N x_N + c_0$$

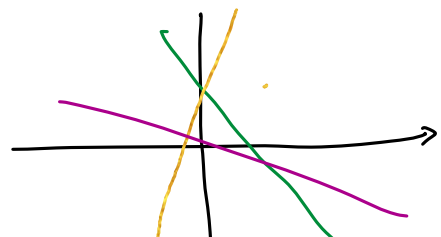
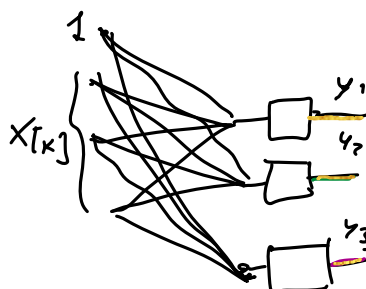


$$d[k] \in \{a_1, a_2, a_3\}$$

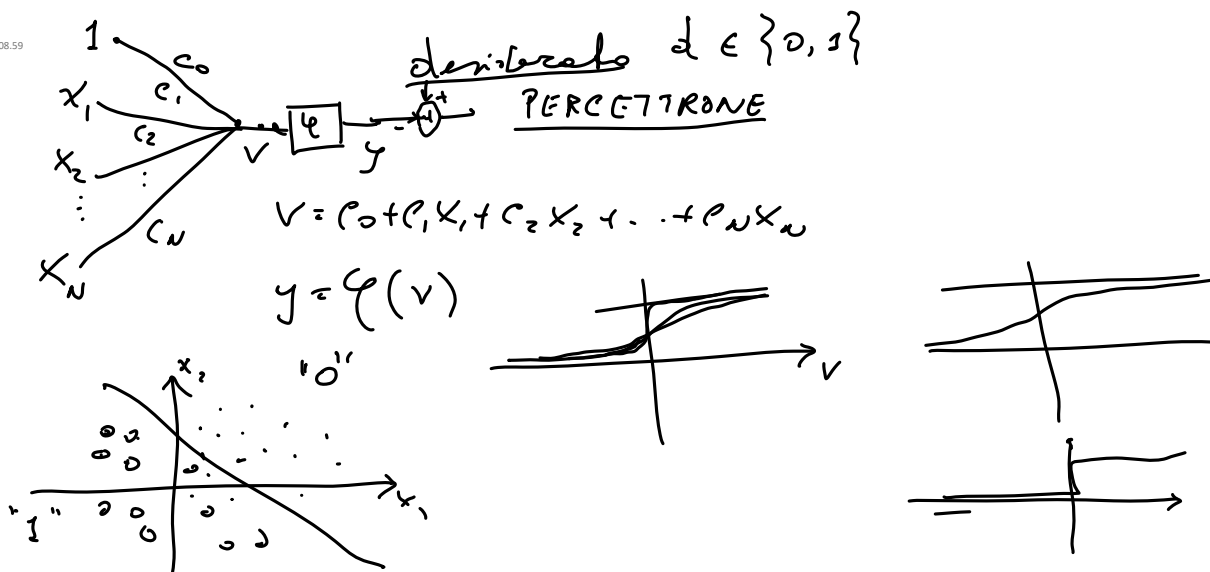


$$\begin{aligned} a_1 &\rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ a_2 &\rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ a_3 &\rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

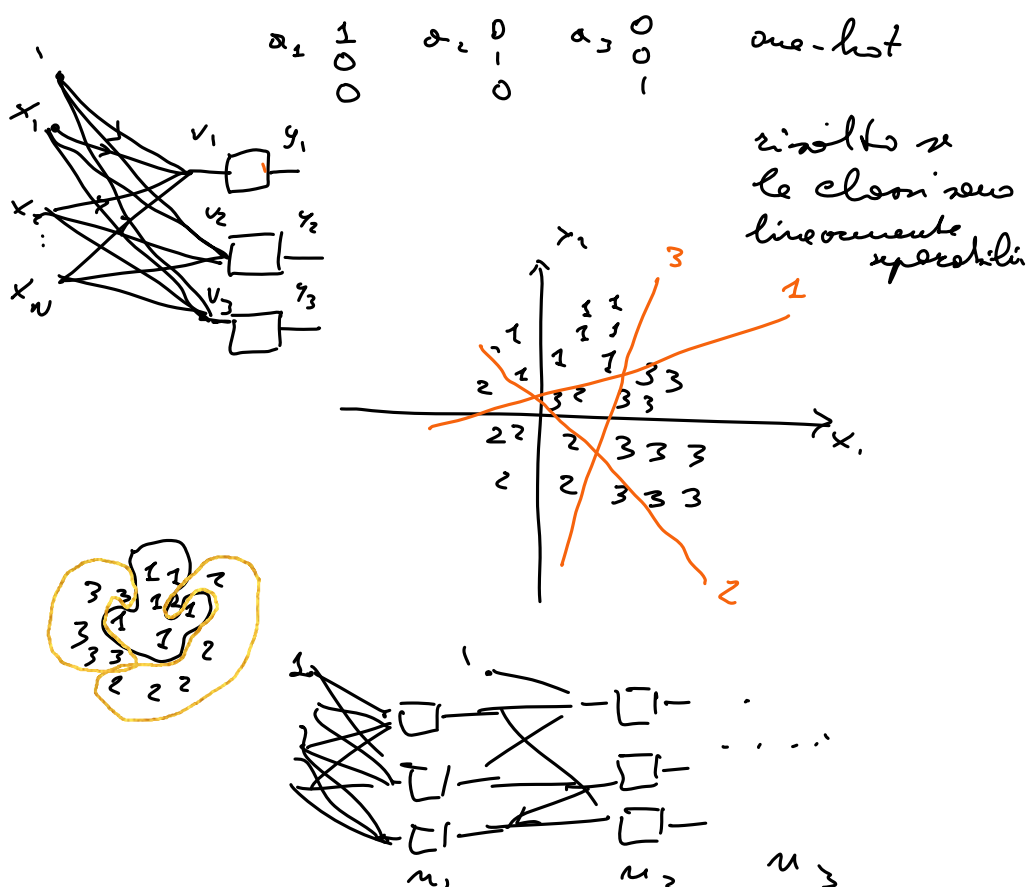
one-hot



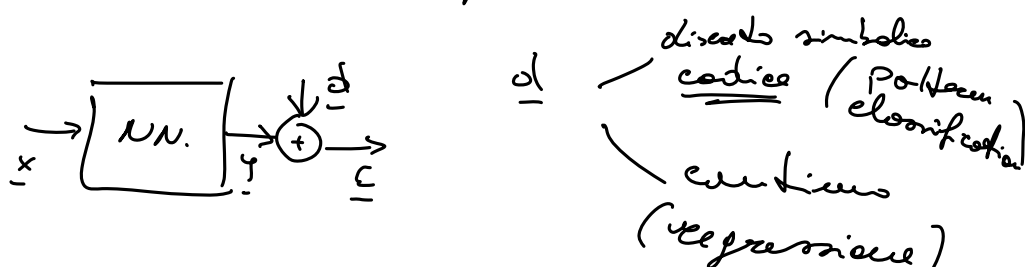




3 classi: $d \in \{a_1, a_2, a_3\}$

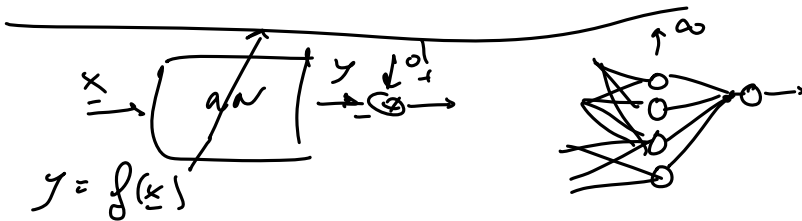
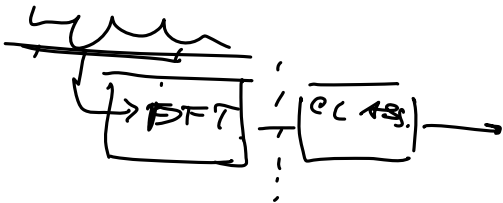
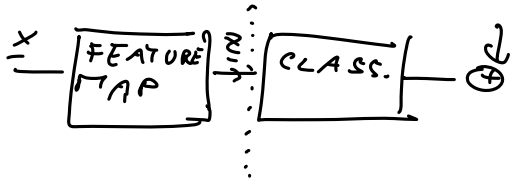


Apprendimento supervisionato (Teacher)
non supervisionato (no teacher)



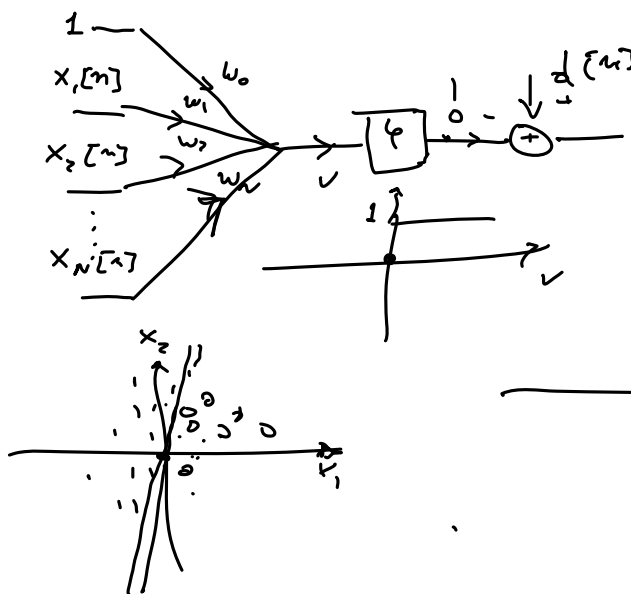
various
(representation)

PATTERN RECOGNITION



ESEMPLO (Training set) $d[i] \in \{0, 1\}$

$\{(x[1], d[1]), (x[2], d[2]), \dots, (x[L], d[L])\} \rightarrow$ TEST SET

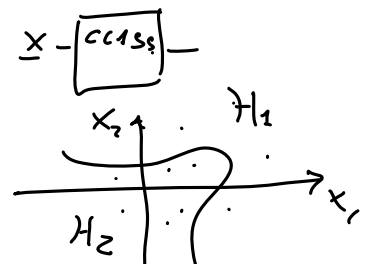


TEST SET
 $x[L+1] \rightarrow \hat{d}[L+1]$
 $x[L+2] \rightarrow \hat{d}[L+2]$
 $\vdots$
 $\rightarrow \hat{d}[L+L]$
 GENERALIZATION

CLASSIFICAZIONE BAYESIANA

(CASO BINARIO) c_1, c_2

$$R = c_{11} p_1 \int p(x|c_1) dx + c_{22} p_2 \int p(x|c_2) dx$$



Rischio di Bayes

$$R = c_{11} p_1 \int_{H_1} p(x|c_1) dx + c_{22} p_2 \int_{H_2} p(x|c_2) dx + c_{21} p_1 \int_{H_2} p(x|c_1) dx + c_{12} p_2 \int_{H_1} p(x|c_2) dx$$

$H = H_1 \cup H_2 = \mathbb{R}^n$
 $H_1 \cap H_2 = \emptyset$

$$= c_{11} p_1 \int_{H_1} p(x|c_1) dx + c_{22} p_2 \int_{H_2} p(x|c_2) dx + c_{21} p_1 \int_{H_2} p(x|c_1) dx + c_{12} p_2 \int_{H_1} p(x|c_2) dx$$

$$R = c_{21} p_1 + c_{22} p_2$$

$$+ \int_{H_1} [p_2(c_{12} - c_{22}) p_X(x|c_2) - p_1(c_{21} - c_{11}) p_X(x|c_1)] dx$$

$$p_1(c_{21} - c_{11}) p(x|c_1) \geq_{H_2} p_2(c_{12} - c_{22}) p(x|c_2)$$

$x \rightarrow \Delta$

$$\frac{p(x|c_1)}{p(x|c_2)} \geq_{H_2} \frac{p_2(c_{12} - c_{22})}{p_1(c_{21} - c_{11})}$$

$\Delta(x)$ RAPPORTO DI VEROSIMILIANZA

c_{22} costo di un errore nel caso $2 \neq 0$
 c_{11} costo di un errore nel caso $1 \neq 0$
 c_{12} costo di errore da $2 \neq 1$
 c_{21} " errore da $1 \neq 2$) $c_{12} = c_{21} =$

$$\frac{p(x|c_1)}{p(x|c_2)} \geq_{H_2} \frac{p_2}{p_1}$$

REGOLA MAP (MAXIMUM A POSTERIORI PROBABILITY)

$$\frac{p(x|c_1) p_1}{p(x)} \geq_{H_2} \frac{p(x|c_2) p_2}{p(x)}$$

Minimizzare Prob. di errore

$$p(c_1|x) \geq_{H_2} p(c_2|x)$$

Prob a posteriori

MAP $\longleftrightarrow$ Min prob. errore

Se $p_1 = p_2$

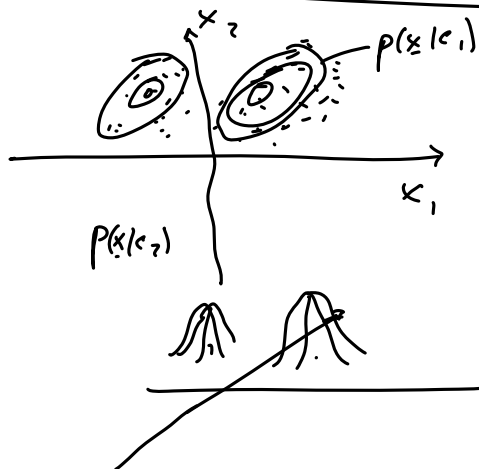
$$p(x|c_1) \geq_{H_1} \dots$$

$$\mu_1 = \mu_2$$

$$\text{verosimiglianza} \rightarrow p(\underline{x}|c_1) \sum_{H_1}^{H_1} p(\underline{x}|c_2)$$

RECOLA A MASSIMA VEROSIMILIANZA

lg



CASO GAUSSIANO

$$p(\underline{x}|c_1) = N(\underline{x}; \underline{\mu}_1, \underline{\Sigma}_1)$$

$$p(\underline{x}|c_2) = N(\underline{x}; \underline{\mu}_2, \underline{\Sigma}_2)$$

$$\underline{\Sigma}_1 = \underline{\Sigma}_2$$

MAP

$$p(\underline{x}|c_1) p_1 \sum_{H_1}^{H_1} p(\underline{x}|c_2) p_2$$

$$\frac{1}{(2\pi)^{d/2} |\underline{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_1)} p_1$$

$$\sum \frac{1}{(2\pi)^{d/2} |\underline{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_2)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_2)} p_2$$

$\downarrow \ln$

$$-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_1) + \ln p_1$$

$$\sum_{H_2}^{H_2} -\frac{1}{2}(\underline{x}-\underline{\mu}_2)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_2) + \ln p_2$$

$$\begin{aligned} & -\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_1) + \frac{1}{2}(\underline{x}-\underline{\mu}_2)^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu}_2) \\ & \sum_{H_1}^{H_1} -2 \ln p_1 + 2 \ln p_2 \end{aligned}$$

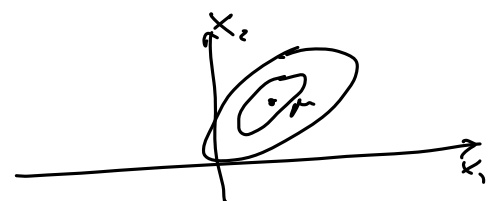
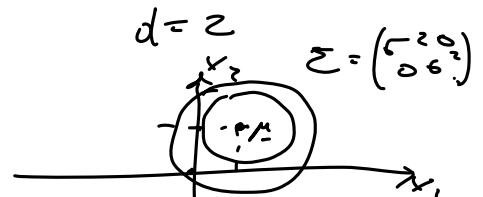
$$\begin{aligned} & -\left[\underline{x}^T \underline{\Sigma}^{-1} \underline{x} + \underline{\mu}_1^T \underline{\Sigma}^{-1} \underline{\mu}_1 - 2 \underline{\mu}_1^T \underline{\Sigma}^{-1} \underline{x} \right] \\ & + \left[\underline{x}^T \underline{\Sigma}^{-1} \underline{x} + \underline{\mu}_2^T \underline{\Sigma}^{-1} \underline{\mu}_2 - 2 \underline{\mu}_2^T \underline{\Sigma}^{-1} \underline{x} \right] \end{aligned}$$

Forma canonica

$$f(\underline{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\underline{x}-\underline{\mu})^2}{2\sigma^2}}$$

$$f(\underline{x}) = \frac{1}{(2\pi)^{d/2} |\underline{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})}$$

$\underline{x} \in \mathbb{R}^d$



$$T [\sqrt{2} x + \mu_1 \geq \mu_2 - \sqrt{2} \mu_2 \geq x]$$

$$\geq -2 \ln \frac{P_2}{P_1}$$

$$2 \mu_1^T \Sigma^{-1} x - 2 \mu_2^T \Sigma^{-1} x \geq \sum_{H_1} \mu_1^T \Sigma^{-1} \mu_1 - \mu_2^T \Sigma^{-1} \mu_2 - 2 \ln \frac{P_2}{P_1}$$

$$(\mu_1 - \mu_2)^T \Sigma^{-1} x \geq \underbrace{\sum_{H_1} \frac{\mu_1^T \Sigma^{-1} \mu_1}{2} - \frac{\mu_2^T \Sigma^{-1} \mu_2}{2}}_{\text{costante}} - \ln \frac{P_2}{P_1}$$

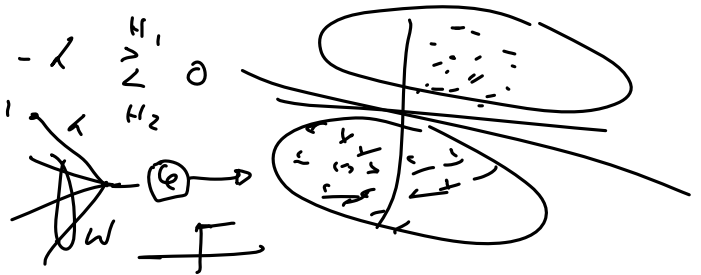
lineare.

$$w^T x \geq \sum_{H_1} 1$$

costante.

$$w^T x - 1 \geq \sum_{H_2} 0$$

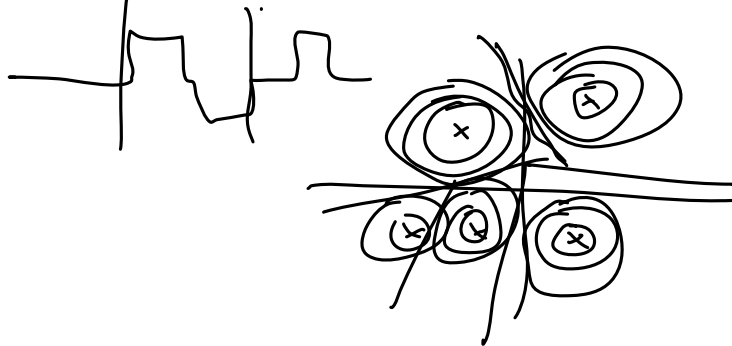
conseguenza a 2 classi: $\Sigma_1 = \Sigma_2$
MAP è class. lineare



Most merca.

Segnali

$\times$ Spazio di input.



$$p(\underline{x}|c_1) \quad p(\underline{x}|c_2)$$

$$p_1(c_{21} - c_{11}) p_X(\underline{x}|c_1) > p_2(c_{12} - c_{22}) p_X(\underline{x}|c_2)$$



$$p(\underline{x}|c_1) = \mathcal{N}(\underline{x}; \underline{\mu}_1, \underline{\Sigma}_1)$$

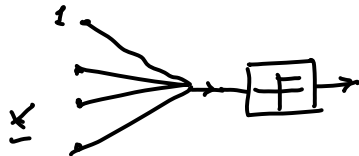
$$p(\underline{x}|c_2) = \mathcal{N}(\underline{x}; \underline{\mu}_2, \underline{\Sigma}_2)$$

$$c_{11} = 0 \quad c_{22} = 0$$

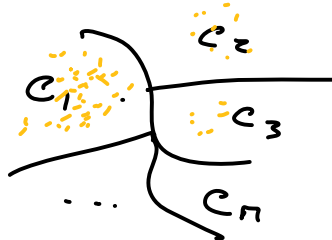
$$c_{12} = c_{21}$$



$$\underline{\Sigma}_1 = \underline{\Sigma}_2 = \sigma^2 \underline{I}$$



$$c_1 \quad c_2 \quad \dots \quad c_M$$



$$p(\underline{x}|c_i)$$

$$p(\underline{x}|c_i) p_i > p(\underline{x}|c_j) p_j$$

$$MAP \equiv \min_{i \neq j} p_i$$

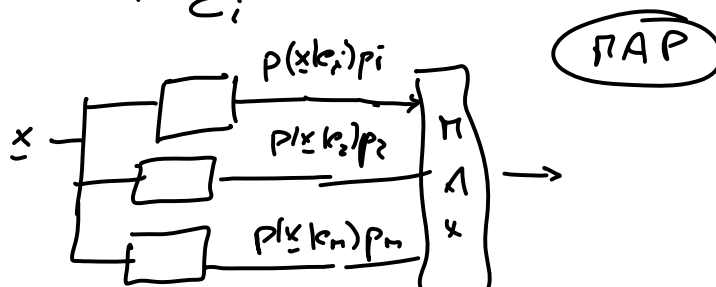
$$\frac{p(\underline{x}|c_i) p_i}{p(\underline{x})} > \frac{p(\underline{x}|c_j) p_j}{p(\underline{x})}$$

$$p(c_i|\underline{x}) > p(c_j|\underline{x})$$

↑ prob. a posteriori

$$p(e) = 1 - p(c) = 1 - \sum_{i=1}^M p(c|c_i) p_i$$

$$= 1 - \sum_{i=1}^M \int p(\underline{x}|c_i) p_i d\underline{x}$$



CASO GAUSSIANO

$$p(\underline{x}|c_i) = \mathcal{N}(\underline{x}; \underline{\mu}_i, \sigma^2 \underline{I})$$

p_i : prob. a priori

$p(\underline{x}|c_i)$ funzione di verosimiglianza

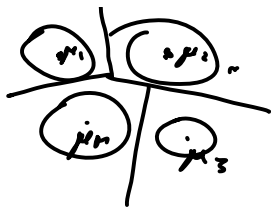
$p(c_i|\underline{x})$ prob. a posteriori

$$p_i = \frac{1}{M} \quad i = 1, \dots, M$$

REGOLA ML

MASSIMA VEROSIMILIANZA

$$p(\underline{x}|c_i) > p(\underline{x}|c_j)$$



$$p(x|c_i) = N(x; \mu_i, \sigma^2 I)$$

$$P(L(p_i = \frac{1}{P_i}))$$

Regola a minima distanza
classifica x con la classe che ha la
media più vicina.

Dimostrazione:

$$\frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1} (x-\mu_i)} > \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu_j)^T \Sigma^{-1} (x-\mu_j)}$$

$$\ln$$

$$-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1} (x-\mu_i) > -\frac{1}{2}(x-\mu_j)^T \Sigma^{-1} (x-\mu_j)$$

$$\Sigma = \sigma^2 I$$

$$-\frac{(x-\mu_i)^T (x-\mu_i)}{\sigma^2} > -\frac{(x-\mu_j)^T (x-\mu_j)}{\sigma^2}$$

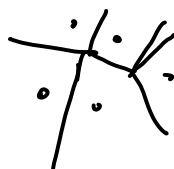
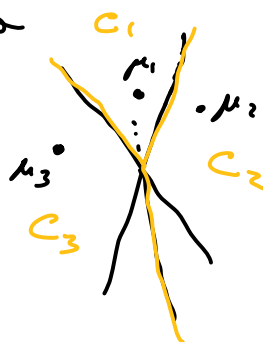
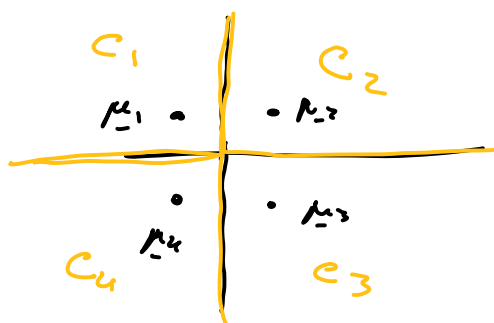
$$\|x - \mu_i\|^2 < \|x - \mu_j\|^2$$

$$d^2(x, \mu_i) < d^2(x, \mu_j) \quad \text{MINIMA DISTANZA.}$$



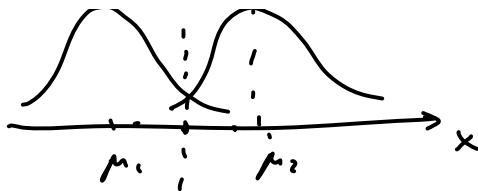
TESSELLAZIONE
DI VORONOI

Regione a minima
distanza



$$d=1 \quad x \quad p(x|c_3) > p(x|c_2)$$





$$\begin{matrix} T & P & U & \rightarrow & \text{Bel tempo } c_1 & 0 \\ & & & & \text{Grande tempo } c_2 & 1 \end{matrix} \quad d \in \{0, 1\}$$

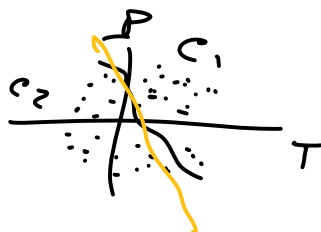
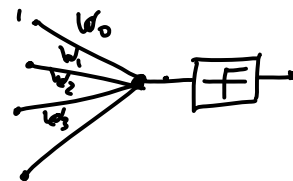
$$\underline{x} = \begin{bmatrix} T \\ P \\ U \end{bmatrix}$$

Se ho $p(x|c_1), p(x|c_2)$
risolvo con la tecnica Bayesiana

Se ho solo degli esempi:

$$\left\{ \begin{array}{l} \underline{x}[1] \underline{x}[2] \dots \underline{x}[L] \\ d[1] \quad d[2] \quad \quad d[L] \end{array} \right\}$$

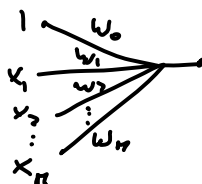
Algoritmo



$$H = \{ (\underline{x}[n], d[n]), n=1, \dots, L \}$$

TRAINING SET

$$d[n] \in \{-1, 1\}$$

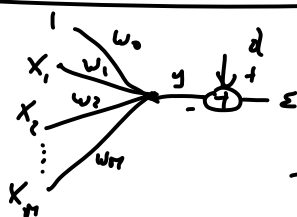


$$J(\underline{w}) = \sum_{n=1}^L (-\underline{w}^T \underline{x}[n] d[n])$$

$$\nabla_{\underline{w}} J = \sum_{n=1}^L (-\underline{x}[n] d[n])$$

$$\underline{w}[n+1] = \underline{w}[n] - \eta[n] \nabla_{\underline{w}} J$$

$$= \underline{w}[n] + \eta[n] \sum_{n=1}^L \underline{x}[n] d[n]$$



$$\{ (\underline{x}[n], d[n]), n=1, \dots, L \}$$

$$J(\underline{w}) = \sum_{n=1}^L \epsilon[n]$$

approssimazione
del minimo
quadrato

Approssimazione
quadratica

$$J(\underline{w}) = E[\epsilon^2[n]]$$

$$\{ (\underline{x}[n], d[n]), n=1, \dots, L \}$$

$$J(\underline{w}) = \sum_{n=1}^L (d[n] - \underline{w}^T \underline{x}[n])^2$$

$$\nabla_{\underline{w}} J = \sum_{n=1}^L 2 (d[n] - \underline{w}^T \underline{x}[n]) (-\underline{x}[n])$$

$$= -2 \sum_{n=1}^L d[n] \underline{x}[n] + 2 \sum_{n=1}^L \underline{w}^T \underline{x}[n] \underline{x}[n] = \underline{0}$$

$$\nabla_{\underline{w}} J = R_x \underline{w} - \underline{r}_{dx}$$

$$\sum_{n=1}^L \underline{x}[n] \underline{x}^T[n] \underline{w}_0 = \sum_{n=1}^L d[n] \underline{x}[n]$$

$$R_x \underline{w}_0 = \underline{r}_{dx}$$

$$\underline{w}_0 = R_x^{-1} \underline{r}_{dx}$$

$$R_x = E[\underline{x} \underline{x}^T]$$

$$\underline{r}_{dx} = E[d \underline{x}]$$

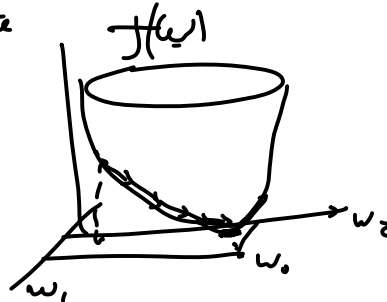
$$R_x \underline{w}_0 = \underline{r}_{dx}$$

Approccio adattativo:

cerchiamo la soluzione mediante un algoritmo al gradiente

$$\underline{w}[k]$$

$$\underline{w}[k] = \underline{w}[k-1] - \eta[k] \nabla_{\underline{w}} J$$



$$\underline{w}[k] = \underline{w}[k-1] - \eta(k) [R_x \underline{w}[k-1] - \underline{r}_{dx}]$$

Newton.

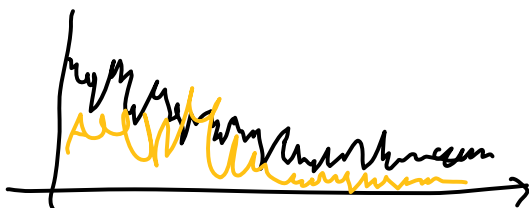
APPROCCIO ISTANTANEO

$$J(\underline{w}, n) = (d[n] - \underline{w}^T \underline{x}[n])^2$$

$$\nabla_{\underline{w}} J(n) = 2 (d[n] - \underline{w}^T \underline{x}[n]) (-\underline{x}[n])$$

$$\underline{w}[n+1] = \underline{w}[n] + \eta(n) \underline{x}[n] (d[n] - \underline{w}[n]^T \underline{x}[n])$$

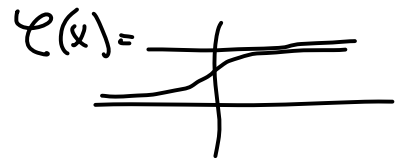
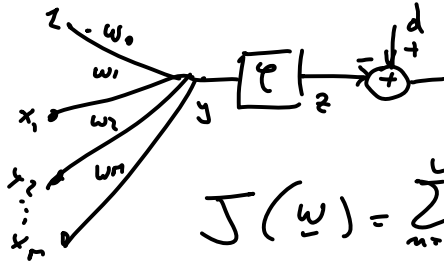
LMS (LEAST MEAN SQUARES)



$$E[\underline{w}[n+1]] = E[\underline{w}[n]] + \eta(n) E[\underline{x}[n] (d[n] - \underline{w}[n]^T \underline{x}[n])]$$

$$E[w[n+1]] = E[w[n]] + \eta[n] E[x[n](d[n] - w_x^T x[n])] \\ = E[w[n]] + \eta[n] (\underline{r}_{d_x} - R_x w[n])$$

APPLICHIAMO L'LMS AL PERCETTORONE



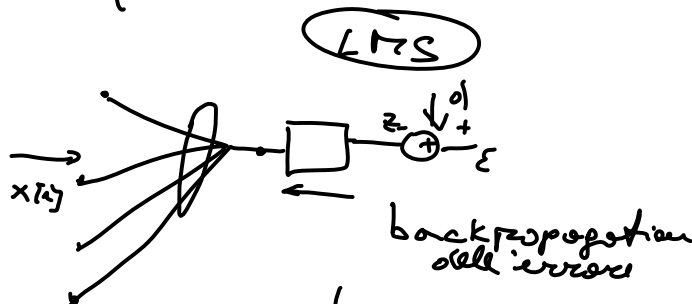
$$J(w) = \sum_{n=1}^L (d[n] - z[n])^2$$

$$\{(\underline{x}[n], d[n]), n=1, \dots, L\} \quad \nabla_w J = \sum_{n=1}^L 2(d[n] - z[n])(-\nabla_w z[n])$$

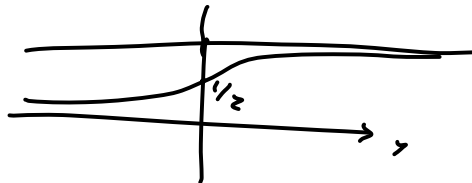
$$\begin{aligned} z &= \varphi(\underline{w}^T \underline{x}) \\ &= \sum_{n=1}^L 2(d[n] - z[n]) \left(-\varphi'(\underline{w}^T \underline{x}[n]) \underline{x}[n] \right) \\ &= -2 \sum_{n=1}^L (d[n] - z[n]) \varphi'(\underline{w}^T \underline{x}[n]) \underline{x}[n] \end{aligned}$$

ADOTTO UNA STRATEGIA ISTANTANEA

$$w[n+1] = w[n] + \eta[n] \underline{x}[n] \varphi'(\underline{w}^T \underline{x}[n]) (d[n] - z[n])$$



$$\varphi(y) = \frac{1}{1 + e^{-\alpha y}}$$



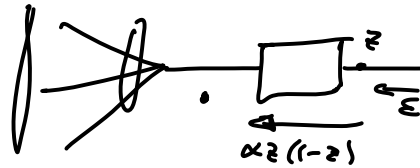
$$\begin{aligned} \varphi'(y) &= \frac{\alpha e^{-\alpha y}}{(1 + e^{-\alpha y})^2} = \\ &= \alpha \frac{e^{\alpha y}}{1 + e^{\alpha y}} \frac{1}{1 + e^{-\alpha y}} \end{aligned}$$

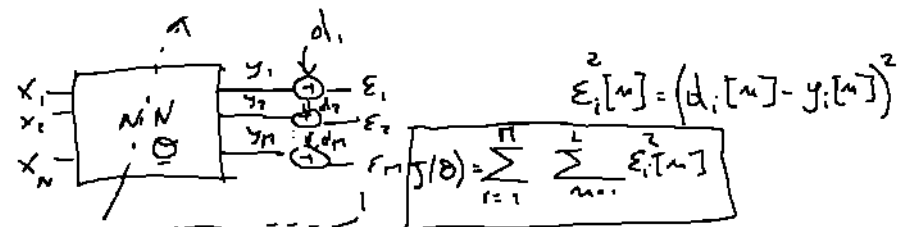
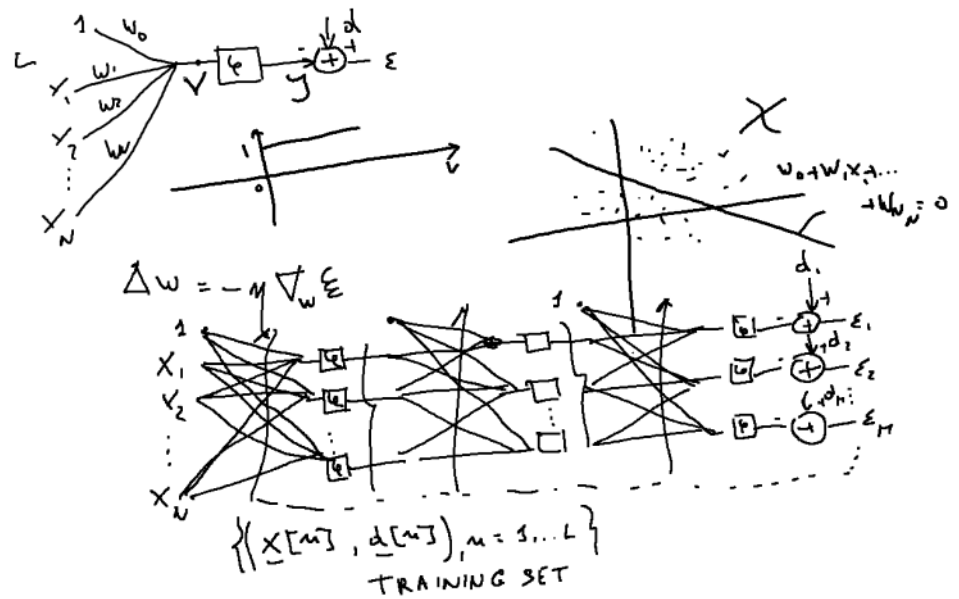
$$\begin{aligned} 1 - \varphi(y) &= 1 - \frac{1}{1 + e^{-\alpha y}} \\ &= \frac{1 + e^{-\alpha y} - 1}{1 + e^{-\alpha y}} \end{aligned}$$

$$\boxed{\varphi'(y) = \alpha (1 - \varphi(y)) \varphi(y)}$$

$$\boxed{\varphi'(y) = \alpha \underbrace{(1 - \varphi(y))}_{1-z} \underbrace{\varphi(y)}_z}$$

$$w[n+1] = w[n] + \gamma[n] \alpha x[n] z[n] (1 - z[n]) (\underbrace{d[n] - z[n]}_{\text{error}})$$



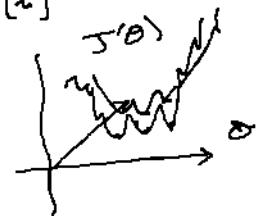


$$\{X[n], d[n], n=1 \dots L\}$$

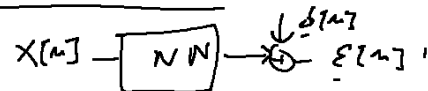
$$\theta_0 = \underset{\theta}{\operatorname{argmin}} J(\theta) = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^M \sum_{n=1}^L \epsilon_i^2[n]$$

a) BATCH LEARNING

b) ON-LINE LEARNING



BATCH LEARNING



$$\{X[n], d[n], n=1, \dots, L\} \rightarrow J(\theta) = \sum_{n=1}^L \sum_{i=1}^I \epsilon_i^2[n]$$

$\forall$ epoch

$$\theta[k+1] = \theta[k] - \eta(k) \nabla_{\theta} J(\theta)$$

EPOCH:



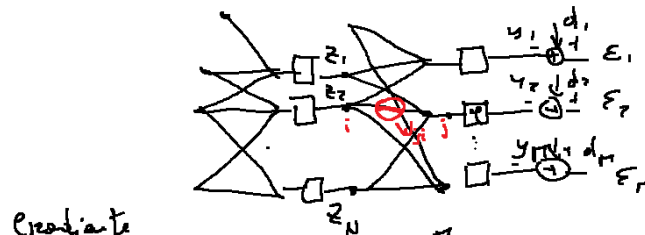
ON-LINE LEARNING

$$\{X[n], d[n]\} \rightarrow J(\theta, n) = \sum_{i=1}^I \epsilon_i^2[n]$$

$$\theta[n+1] = \theta[n] - \eta(n) \nabla_{\theta} J(\theta, n)$$

$\forall$ esempio

Aggiornamento
per ogni singolo
esempio



Gradiente
istantaneo

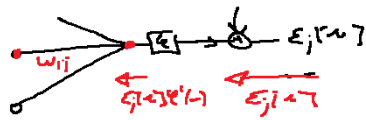
$$J(n) = \sum_{j=1}^I \epsilon_j^2[n] = \sum_{j=1}^I (d_j[n] - y_j[n])^2$$

$$= \sum_{j=1}^I (d_j[n] - \phi(\sum_{i=1}^N w_{ji} z_i[n]))^2$$

$$\frac{\partial J(n)}{\partial w_{ij}} = \frac{\partial \epsilon_j^2[n]}{\partial w_{ij}} = -2(d_j[n] - y_j[n]) \frac{\partial y_j[n]}{\partial w_{ij}}$$

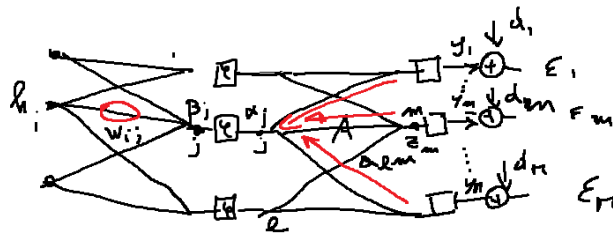
$$= -2 \underbrace{(d_j[n] - y_j[n])}_{\epsilon_j[n]} \phi'(\sum_{i=1}^N w_{ji} z_i[n]) z_i[n]$$

$$w_{ij}[n+1] = w_{ij}[n] + \eta(n) z_i[n] \epsilon_j[n] \phi'(\sum_{i=1}^N w_{ji} z_i[n])$$

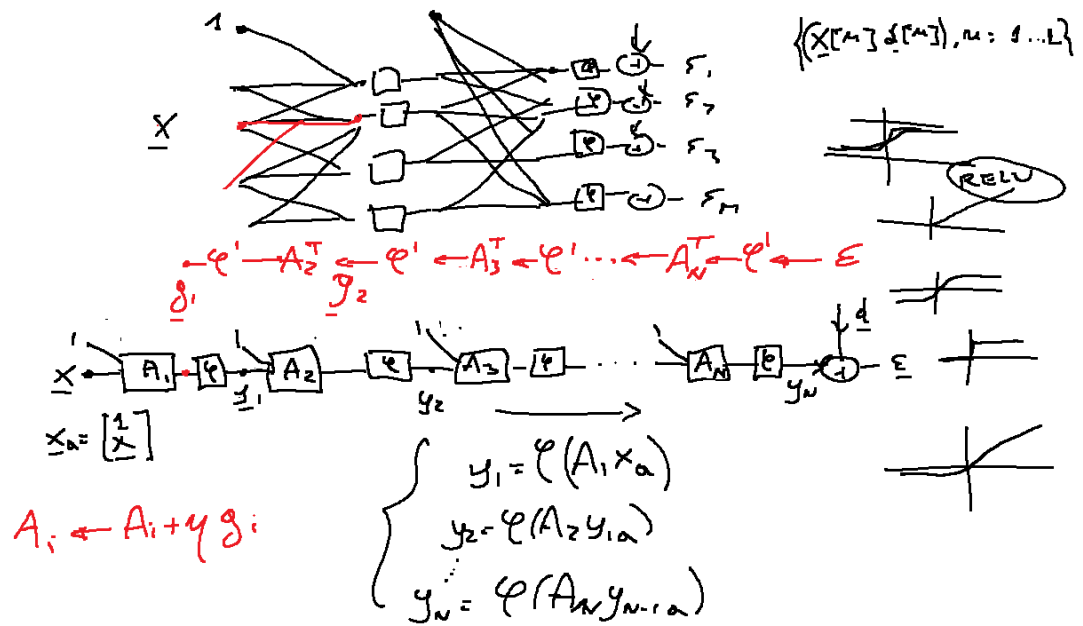


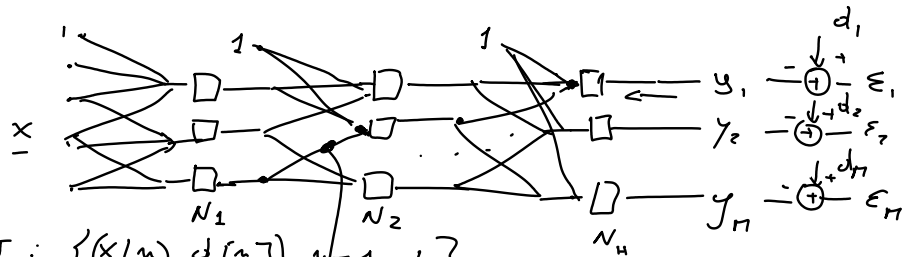
$$\Delta w_{ij} = \eta \delta_i \cdot \epsilon_{\text{backpropagato}}$$

1X



$$\begin{aligned} \frac{\partial \sum_{n=1}^M \epsilon_n^2}{\partial w_{ij}} &= 2 \sum_{n=1}^M \frac{\partial \epsilon_n}{\partial w_{ij}} \epsilon_n \\ &= 2 \sum_{n=1}^M \epsilon_n \frac{\partial \epsilon_n}{\partial z_n} \frac{\partial z_n}{\partial w_{ij}} = 2 \sum_{n=1}^M \epsilon_n \varphi'(z_n) \frac{\partial z_n}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial w_{ij}} \\ &= -2 \sum_{n=1}^M \epsilon_n \varphi'(z_n) \alpha_{ij} \frac{\partial \alpha_j}{\partial \beta_j} \frac{\partial \beta_j}{\partial w_{ij}} \\ &= -2 \sum_{n=1}^M \epsilon_n \varphi'(z_n) \alpha_{ij} \varphi'(\beta_j) h_i \end{aligned}$$





TRAINING SET: $\{(x^{(n)}, d^{(n)}), n=1 \dots L\}$

ALG. ON-LINE $(x^{(n)}, d^{(n)})$

- Propagazione di $x^{(n)}$ in avanti.
- Errori in uscita
- Calcolo del gradiente

$$w[n+1] = w[n] - \nabla_{w[n]} E(n)$$

$$E(n) = \sum_{i=1}^n \epsilon_i^2[n]$$

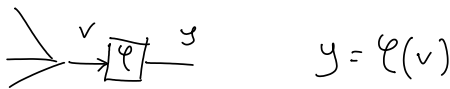
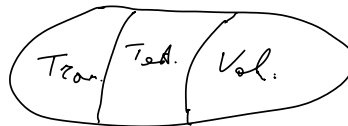
$\nabla_w E$ calcolato mediante l'alg. di "back propagation"

EPOCH (EPOCHI)

GENERALIZZAZIONE

TEST SET $\left\{ \begin{matrix} x^{(n)} \\ d^{(n)} \end{matrix} \right\}, n=L+1 \dots L+T \rightarrow y^{(n)}$

VALIDATION SET



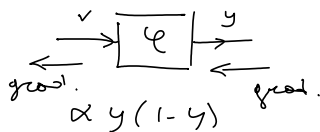
Funzione logistica



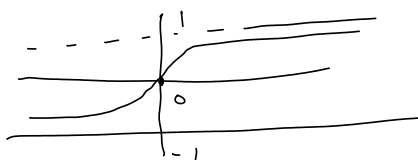
$$\phi(v) = \frac{1}{1 + e^{-\alpha v}} \quad \alpha > 0 \quad \text{differenziabile}$$

$$\phi'(v) = \frac{+e^{-\alpha v} \alpha}{(1 + e^{-\alpha v})^2} = \alpha \phi(v)(1 - \phi(v)) \quad 1 - \phi(v) = 1 - \frac{1}{1 + e^{-\alpha v}}$$

$$= \frac{1 + e^{-\alpha v} - 1}{1 + e^{-\alpha v}} = \frac{e^{-\alpha v}}{1 + e^{-\alpha v}}$$



$$\phi(v) = \frac{\phi(v) + 1}{2}$$



$$\phi(v) = 2\left(\phi(v) - \frac{1}{2}\right) = 2\phi(v) - 1$$

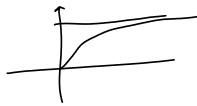
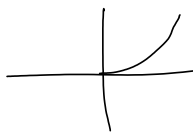
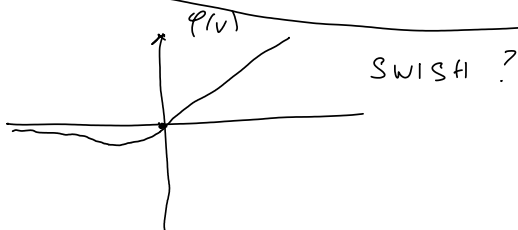
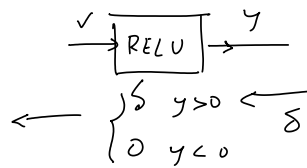
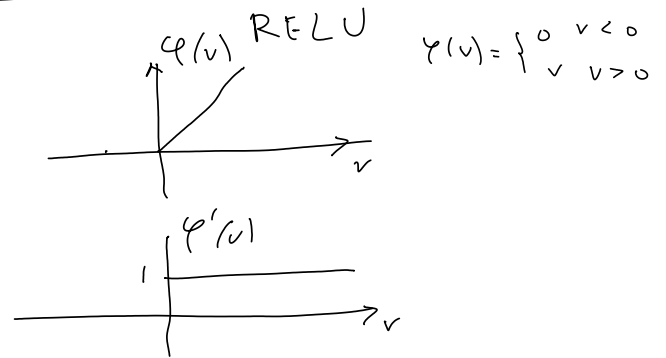
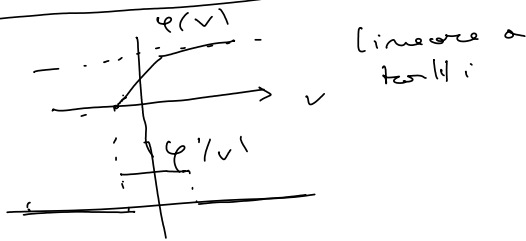
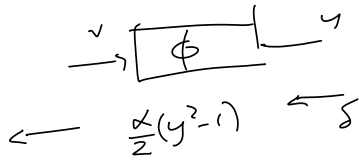
$$= 2 \frac{1}{1 + e^{-\alpha v}} - \frac{1}{2} = \frac{2 - 1 + e^{-\alpha v}}{2(1 + e^{-\alpha v})}$$

$$= \frac{1}{2} \frac{1 - e^{-\alpha v}}{1 + e^{-\alpha v}} = \frac{1}{2} \frac{e^{-\frac{\alpha v}{2}} - e^{\frac{\alpha v}{2}}}{e^{\frac{\alpha v}{2}} + e^{-\frac{\alpha v}{2}}} = \frac{1}{2} \logh \frac{\alpha v}{2}$$

$$\phi(v) = \frac{1}{2} \logh \frac{\alpha v}{2}$$

$$\phi'(v) = 2 \phi'(v) = 2 \alpha \phi(v)(1 - \phi(v))$$

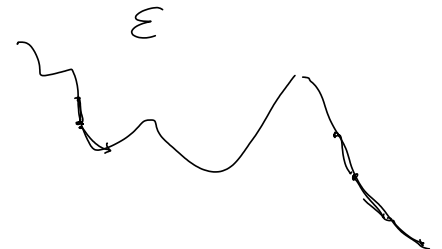
$$\begin{aligned}\phi'(v) &= 2 \quad \varphi'(v) = 2 \alpha \varphi(v) (1 - \varphi(v)) \\ &= 2 \alpha \frac{\phi(v)+1}{2} \left(1 - \frac{\phi(v)+1}{2}\right) = (\phi(v)+1) \alpha \frac{2 - \phi(v) - 1}{2} \\ &= (\phi(v)+1) \frac{(1 - \phi(v))}{2} \alpha = \frac{\alpha}{2} (\phi(v)^2 - 1)\end{aligned}$$



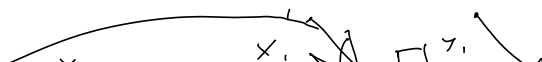
ADAPTATIVNO ζ K MOMENTUM

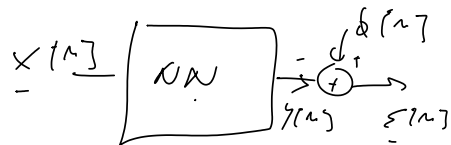
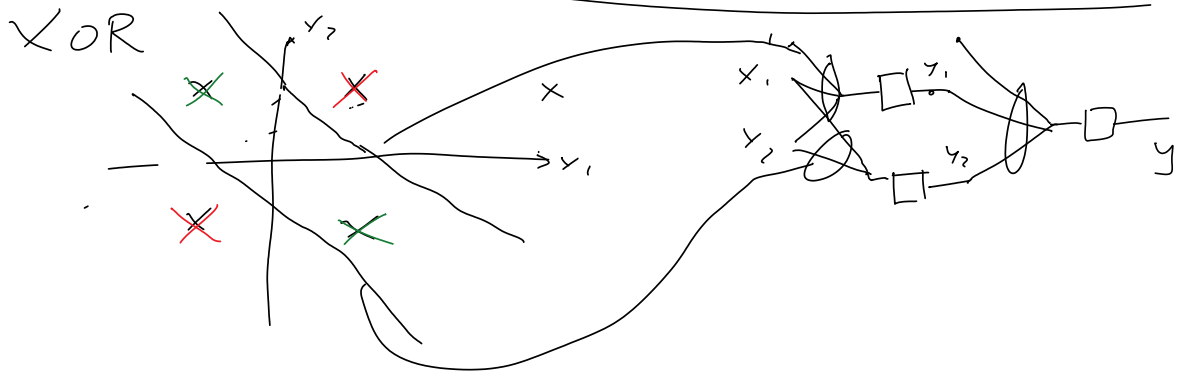
$$\Delta w[n] = \alpha \Delta w[n-1] + \eta \sum \delta_i x_i[n]$$

ADAM $\rightarrow$ MOMENTUM



XOR





BATCH : Presentare tutto il training set
 $\{x[n], d[n], n=1, \dots, L\}$ e calcolare il
 gradiente

$$E = \sum_{n=1}^L \sum_{i=1}^M \epsilon_i^2[n]$$

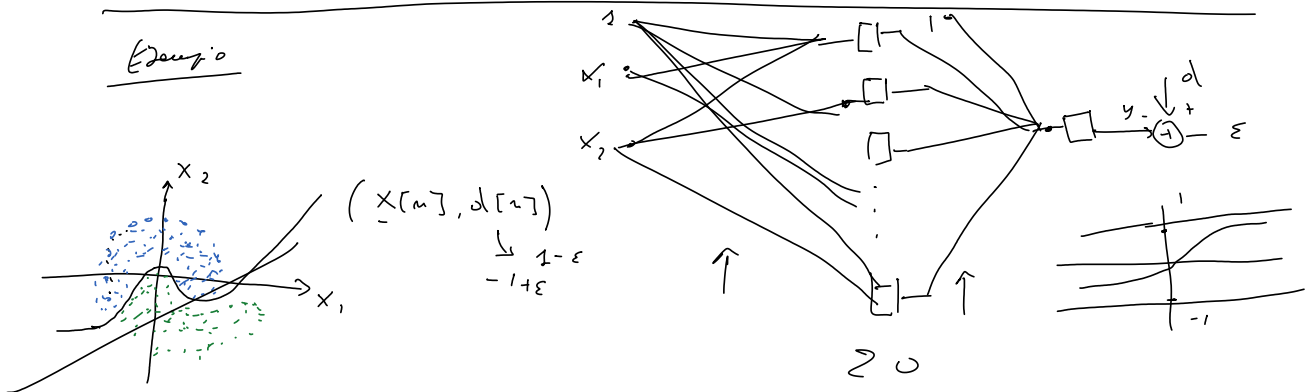
$$\nabla_w E$$

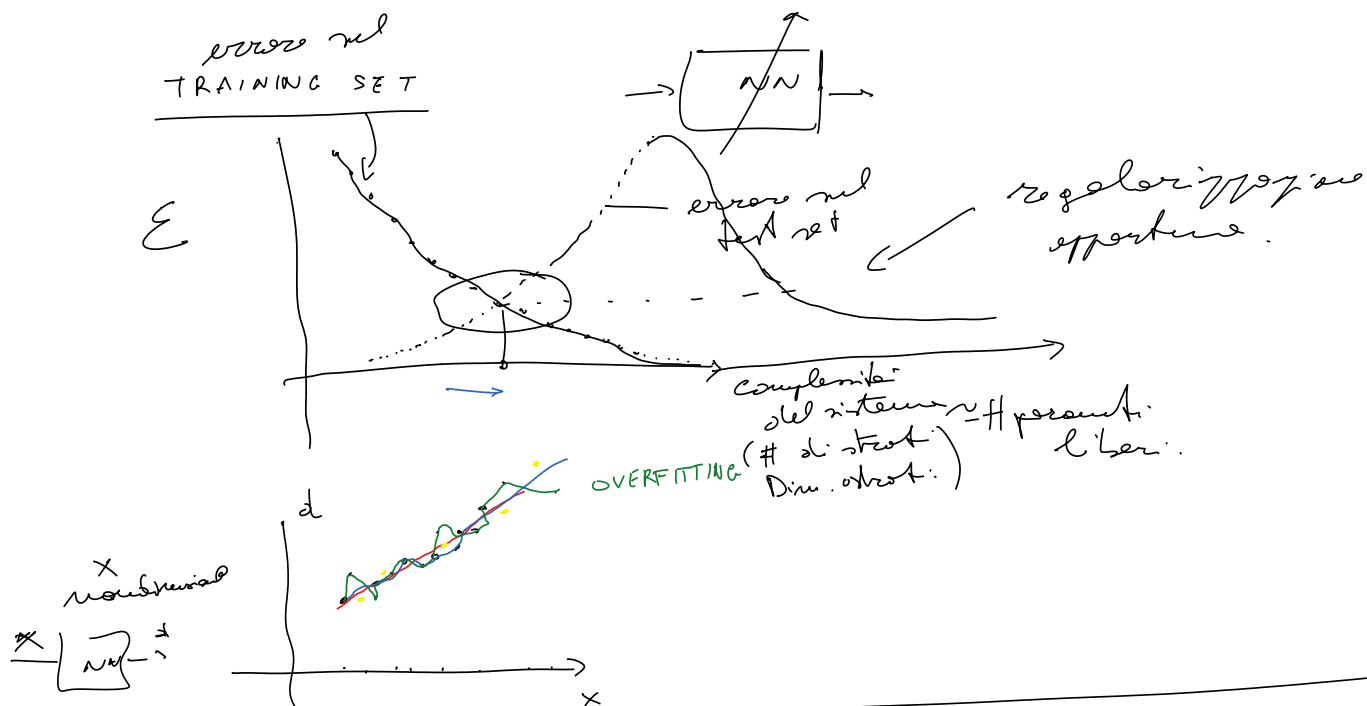
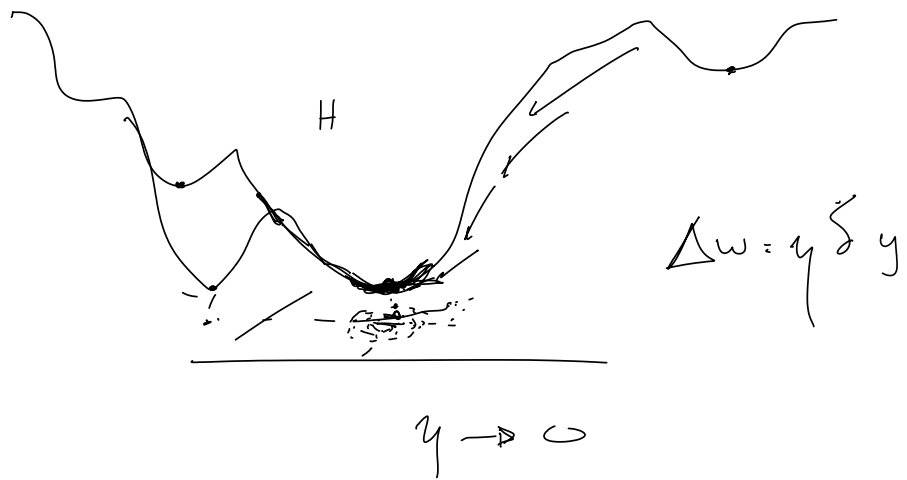
Localmente il gradiente
 è il gradiente medio

$\Rightarrow$ 1 passo di aggiornamento.

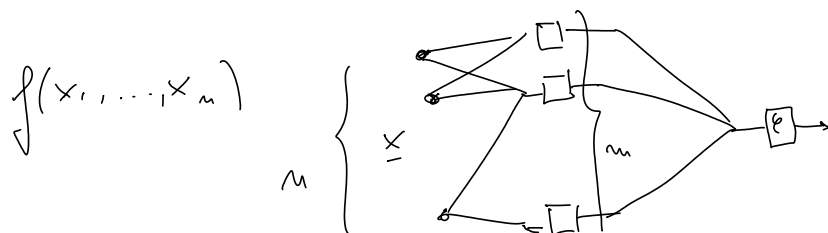
STOCASTICO $\Rightarrow$ 1 esempio alla volta
 o a volte ogni volta

Esempio





TEOREMA (RAPPRESENTAZIONE)

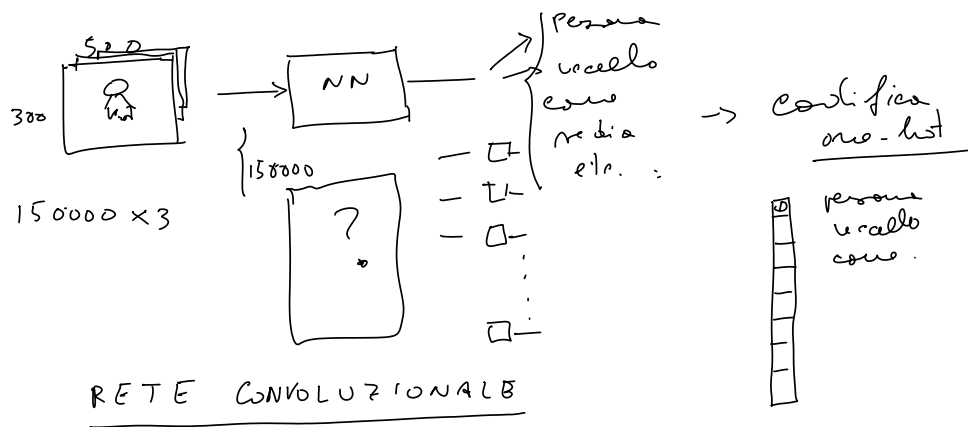


$$|f(x_1, \dots, x_n) - F(x_1, \dots, x_n)| < \epsilon$$

$$\forall \epsilon > 0 \quad \exists m_0 : m > m_0$$

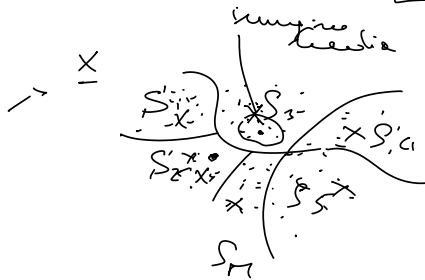
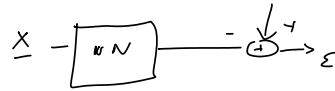
$$0 < \epsilon < 1$$

Riconoscimento di oggetti

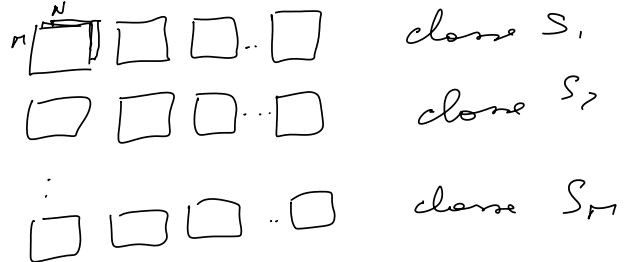


PATTERN RECOGNITION

$x \rightarrow \text{classe } d \in \{c_1, c_2, \dots, c_M\}$



$\{x[k], d[k]\}$

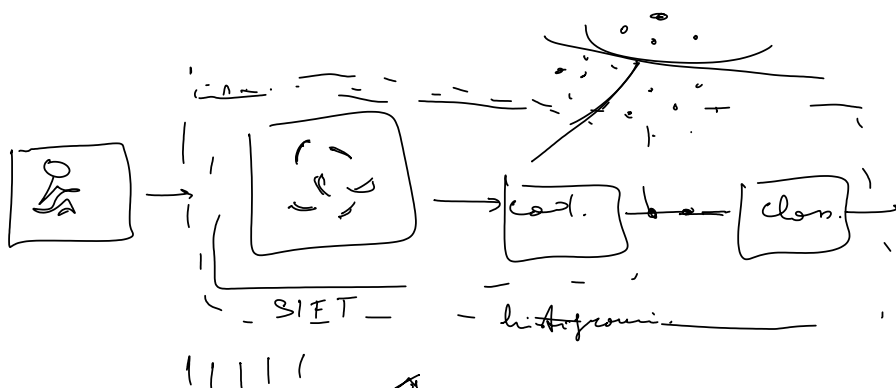
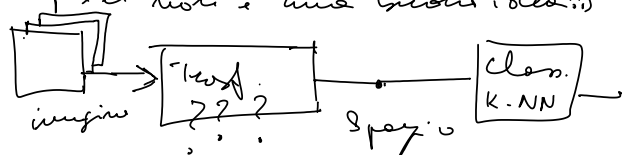


- ① Calcola media di ogni classe
- ② Associa all'immagine di test la classe con la media più vicina.

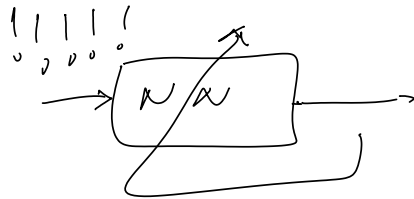
- ① Memorizzo tutto il dataset
- ② Confronto l'immagine di test con tutto il dataset e associa la classe con lo "score" massimo
- ① Memorizzo un set "tipico" per ogni classe (prototipi)
- ② Confronto l'immagine di test con i prototipi e assegna la classe con il migliore "score".

K-nearest neighbour: l'immagine di test trovando le immagini più simili

NON FUNZIONA SULLE IMMAGINI DIRETTAMENTE !!
(confronto pixel non è una buona idea!!)

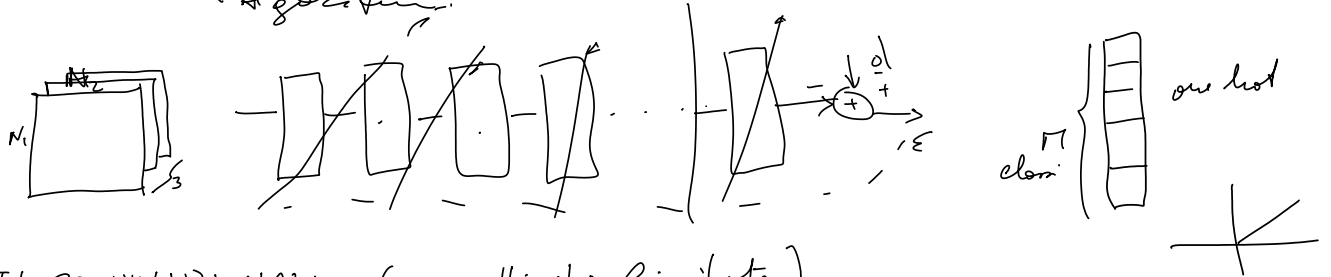


SIFT - histogrammi

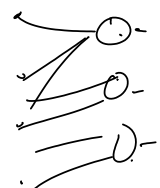
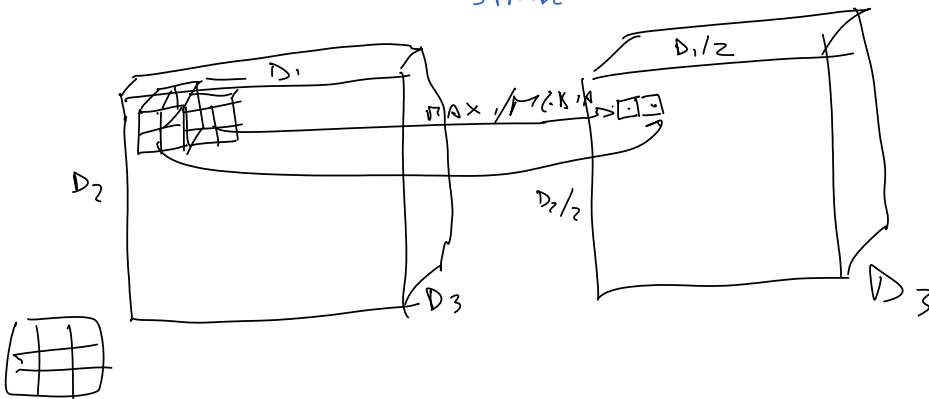
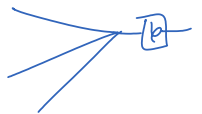
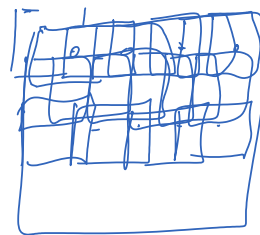
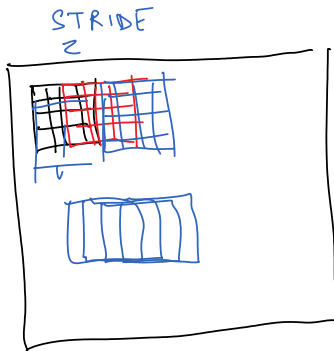
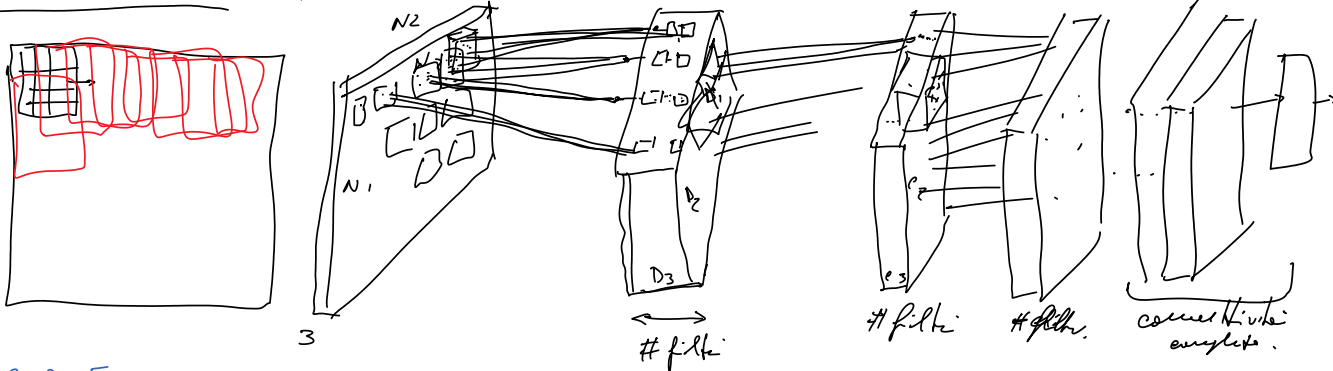


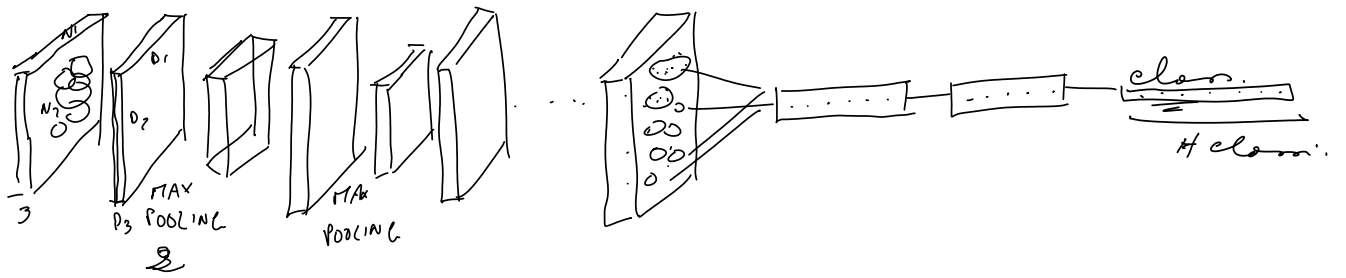
Quale struttura per una NN che realizza un classificatore di immagini??

PROBLEMI: { Disponibilità di dataset di grandi dimensioni
Disponibilità hardware veloce a basso costo (schede grafiche)
Algoritmi.



RETI CONVOLUZIONALI (connessioni limitate)

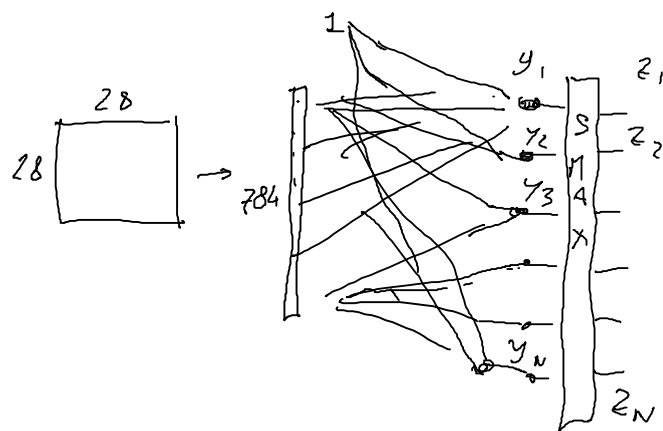




Per visualizzare vedi Deep Visualization Toolbox by Jason Yosinsky su Youtube

(8 Dic 9-11; 11 Dic 9-11; 15 Dic 9-11)

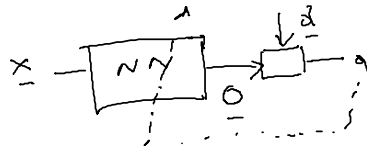
Lezioni di esercitazione su Reti neurali usando Python/Tensor Flow/Keras
Tenute dal Dott. Giovanni Di Gennaro



$$z_i = \frac{y_i}{\sum_{e=1}^N e^{y_e}}$$

SOFTMAX

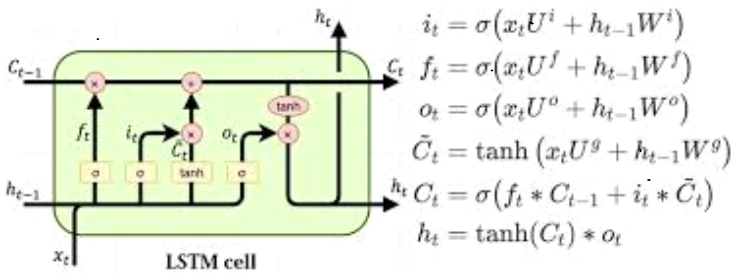
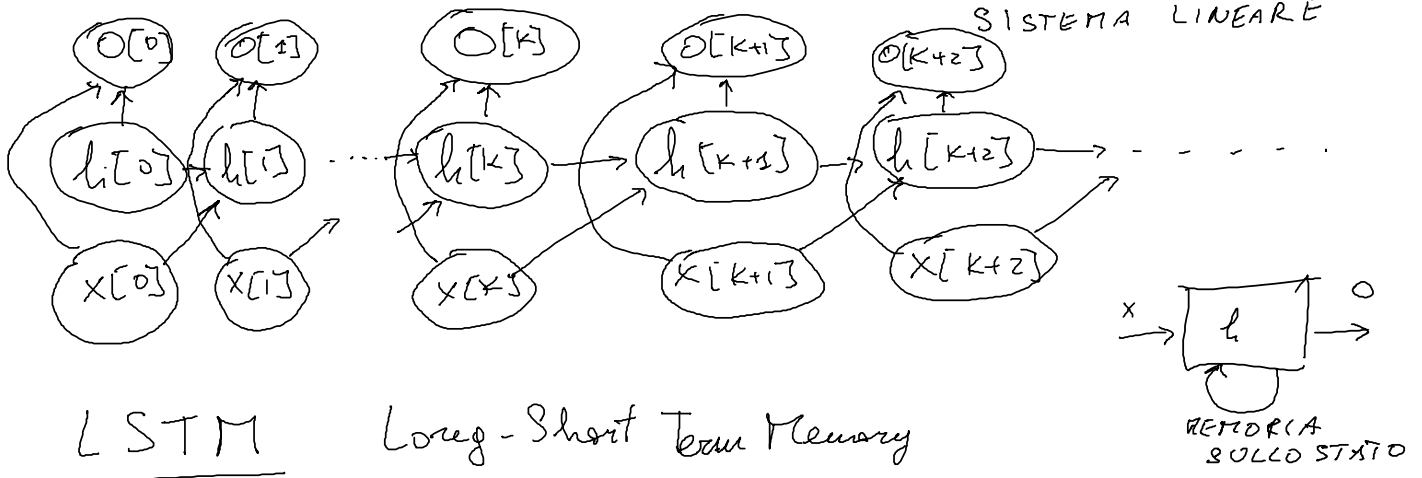
$$0 < z_i < 1$$



$$\begin{cases} \underline{h}[k+1] = f(\underline{h}[k], \underline{x}[k]) \\ \underline{o}[k] = g(\underline{h}[k], \underline{x}[k]) \end{cases}$$

Info Hidden:

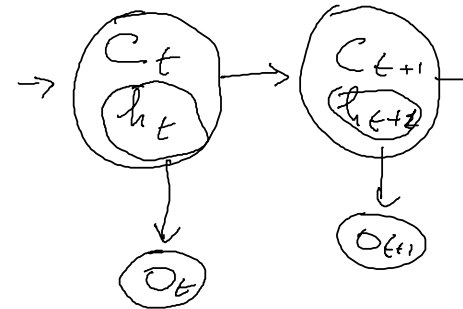
$$\begin{cases} \underline{h}[k+1] = \underline{A} \underline{h}[k] + \underline{B} \underline{x}[k] \\ \underline{o}[k] = \underline{C} \underline{h}[k] + \underline{D} \underline{x}[k] \end{cases}$$



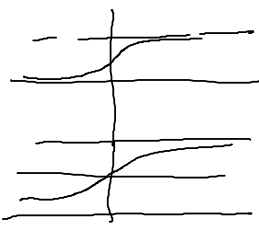
$$\begin{aligned} i_t &= \sigma(x_t U^i + h_{t-1} W^i) \\ f_t &= \sigma(x_t U^f + h_{t-1} W^f) \\ o_t &= \sigma(x_t U^o + h_{t-1} W^o) \\ \tilde{C}_t &= \tanh(x_t U^g + h_{t-1} W^g) \\ C_t &= \sigma(f_t * C_{t-1} + i_t * \tilde{C}_t) \\ h_t &= \tanh(C_t) * o_t \end{aligned}$$

$$\begin{bmatrix} C_t \\ h_t \end{bmatrix}$$

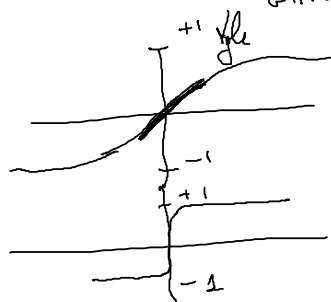
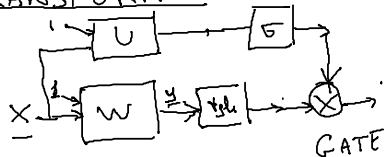
stato



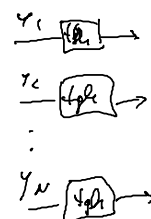
$$C_t = \sigma(f_t \odot C_{t-1} + i_t \cdot \tilde{C}_t)$$



GATED TRANSFORM



$$y = Wx + b$$



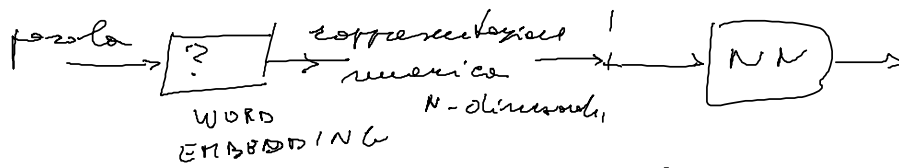
Oggi sono andato al mare } INTENT parole e ASCII (128)
 } RISPOSTA parole e Vocabolario (>60000 elements)

CHATBOT (Leila)

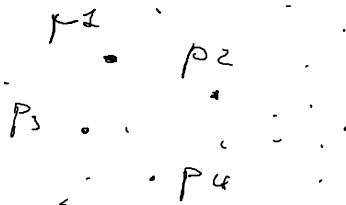
NLP

Natural Language Processing

fino a che era statico:
 posso cambiare l'output

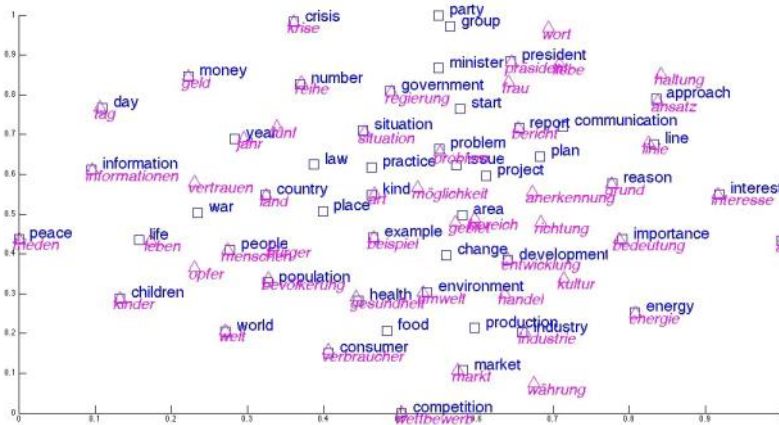


raggruppare
 parole con "significati simili"
 ovvero appartenenti
 allo stesso contesto



cosa
 • pizza
 • usata

come
 • gatto : felino
 • cane



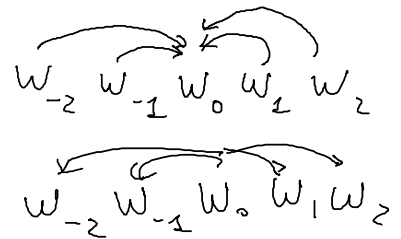
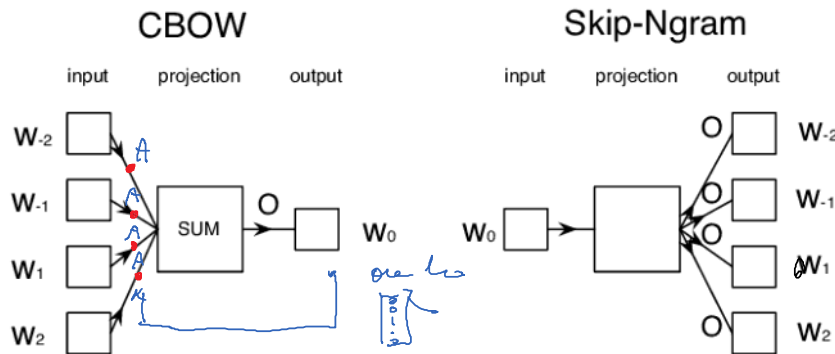
6

POSTULATO:

Parole che copiano vicine in un test appartengono allo stesso "contesto".

Utilizzo di corpora di testi di diverso tipo e associare parole a una rappresentazione vettoriale che raffi a parole che compaiono nella stessa frase proprio vicine.

$\underline{w}_i =$
one-hot



A esibirsi
l'insieme dei
vettori costituenti parole

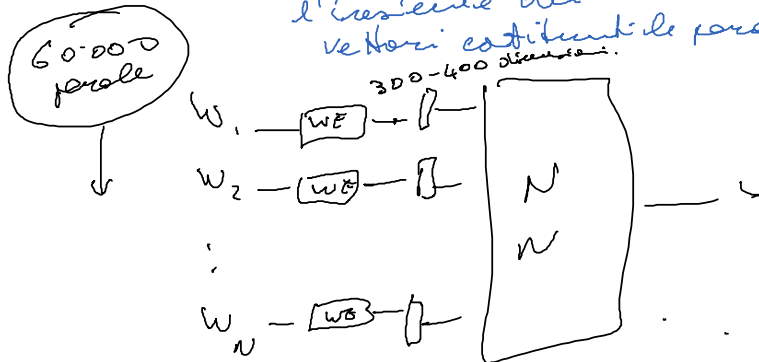
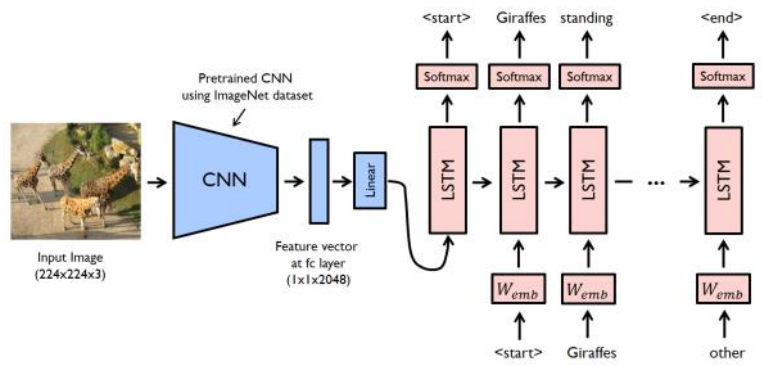


IMAGE CAPTIONING



NIC: a bus that is **sitting** in the street.
ours: a **red and white** bus driving down a street.



NIC: a close up of a **toaster on a wall**.
ours: a close up of a pair of scissors.



NIC: a white plate topped with a cut in half sandwich.
ours: a white plate topped with a sandwich and **salad**.



NIC: a group of people standing on top of a sandy beach.
ours: a group of people standing on a beach with **surfboards**.



NIC: a city street filled with lots of traffic.
ours: a **bus** driving down a street **next to** a traffic light.



NIC: a **person** laying on a bed with a laptop.
ours: a dog laying on a bed in a bedroom.



NIC: a **group of** giraffes standing in a field.
ours: a giraffe standing in a fenced in area.



NIC: a train traveling down tracks next to a forest.
ours: a train is traveling down the tracks **in the snow**.



NIC: a **man** in a field with a frisbee.
ours: a couple of men playing frisbee in a field.



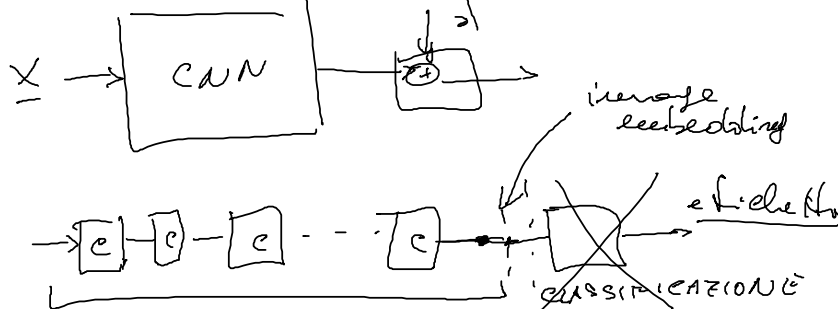
NIC: a man standing **next to** a brown horse.
ours: a man riding a horse in a field.



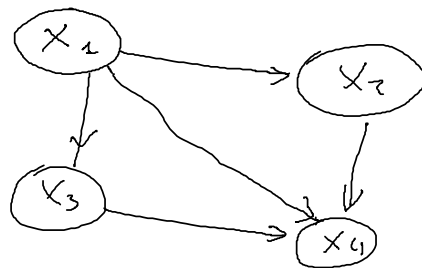
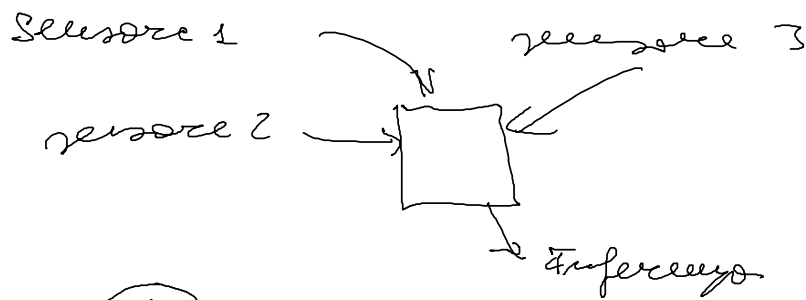
NIC: a **group of people riding bikes** down a street.
ours: a man riding a bike down a busy street.



NIC: a black and white dog **laying on a grass** covered field.
ours: a black and white dog playing with a frisbee.



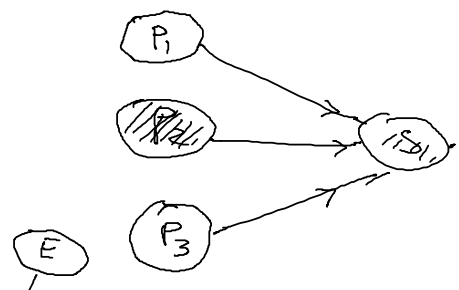
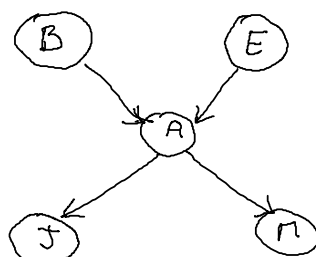
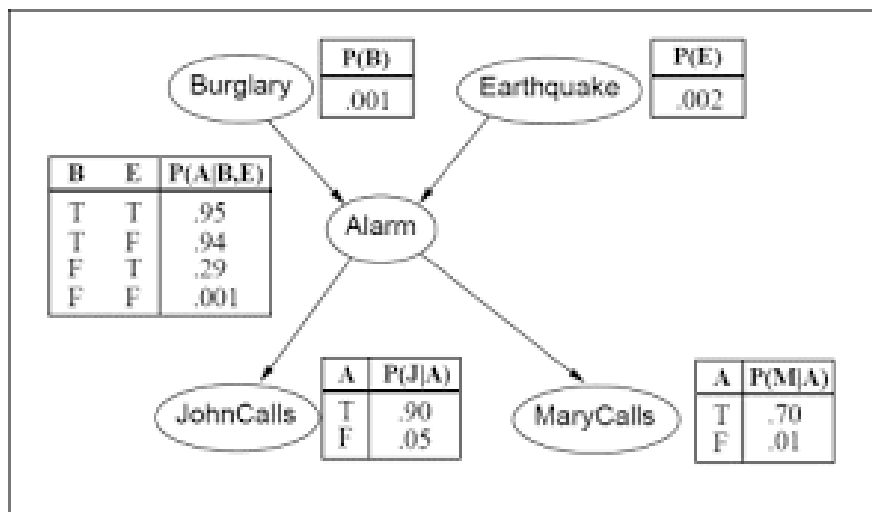
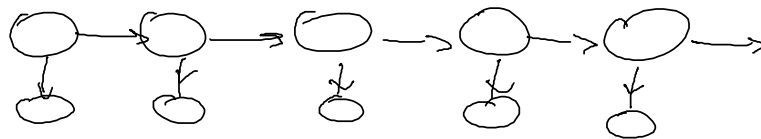
DATA FUSION

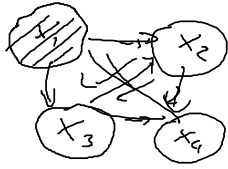
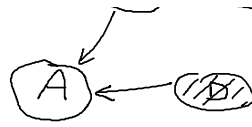


GRAFO BAYESIANO



HIDDEN MARKOV MODEL



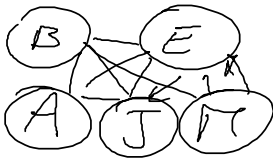


$$p(x_1, x_2, x_3, x_4)$$



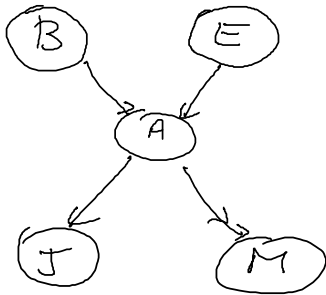
QUERY: $x_1 = \bar{x}_1$ $P(x_2 | x_1 = \bar{x}_1) \propto \frac{P(x_2, x_1 = \bar{x}_1)}{\sum_{x_1} P(x_1 = \bar{x}_1)}$

$$= \sum_{\substack{x_3 \in \mathcal{X}_3 \\ x_4 \in \mathcal{X}_4}} p(\bar{x}_1, x_2, x_3, x_4) \quad \text{Marginalizzazione}$$



"SPAGHETTI"

troppo complesso se non ho informazioni più specifiche sulle relazioni tra le variabili

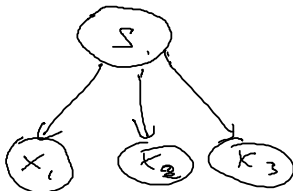


$$p(b, e, a, j, m)$$

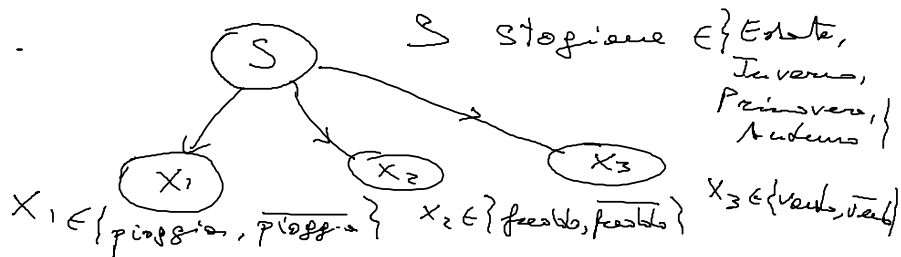
$$= p_E(e) p_B(b) p_{A|BE}(a) p_{J|A}(j) p_{M|A}(m)$$

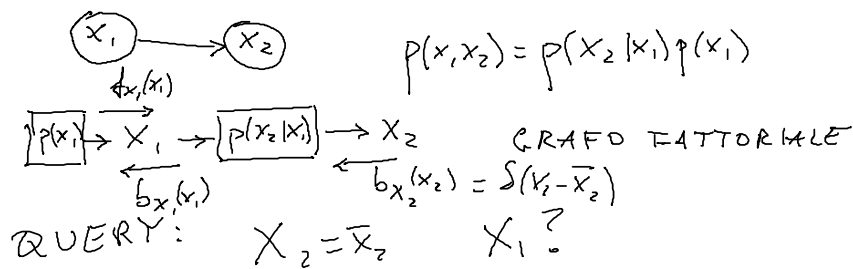


$$p(x_1, x_2, x_3, x_4) = p(x_1) p(x_2 | x_1) p(x_3 | x_2) p(x_4 | x_3)$$



$$p(s, x_1, x_2, x_3) = p(s) p(x_1 | s) p(x_2 | s) p(x_3 | s)$$





$$p(x_1 | X_2 = \bar{x}_2) \propto p(x_1, X_2 = \bar{x}_2) = \underbrace{p(\bar{x}_2 | x_1)}_{b_{x_1}(x_1)} \underbrace{p(x_1)}_{f_{x_1}(x_1)} \quad \leftarrow \text{PRODUCT}$$

POSTERIOR

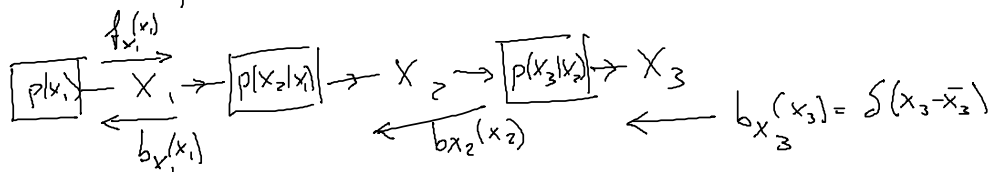
$$p(\bar{x}_2 | x_1) = \sum_{x_2} p(x_2 | x_1) \underbrace{\delta(x_2 - \bar{x}_2)}_{b_{x_2}(x_2)}$$

SUM



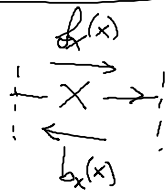
QUERY: $p(x_1 | X_3 = \bar{x}_3) = ?$

$$p(x_1, x_2, x_3) = p(x_1)p(x_2|x_1)p(x_3|x_2)$$

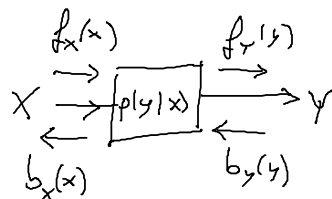


$$p(x_1 | X_3 = \bar{x}_3) \propto p(x_1, X_3 = \bar{x}_3) = \sum_{x_2} p(x_1, x_2, \bar{x}_3) = \sum_{x_2} p(\bar{x}_3 | x_2) p(x_2 | x_1) p(x_1)$$

$$= \underbrace{p(x_1)}_{f_{x_1}(x_1)} \underbrace{\sum_{x_2} p(\bar{x}_3 | x_2) p(x_2 | x_1)}_{b_{x_1}(x_1)}$$

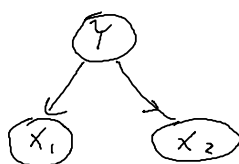


$$p(x | \text{un.}) = f_x(x) b_x(x) \quad \text{POSTERIOR}$$

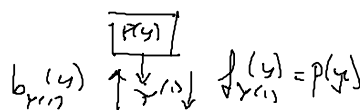


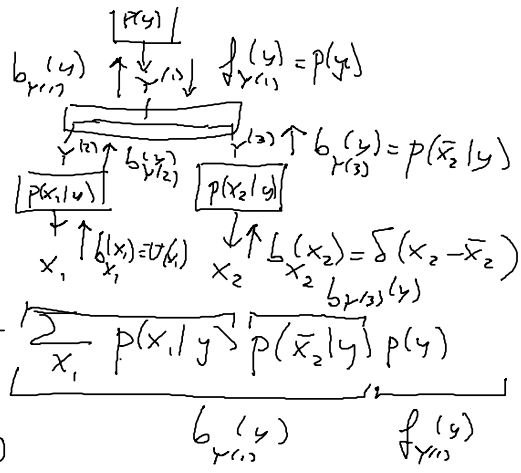
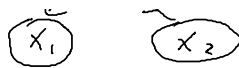
$$f_y(y) = \sum_x p(y|x) f_x(x)$$

$$b_x(x) = \sum_y p(y|x) b_y(y)$$



$$p(y, x_1, x_2) = p(x_1 | y) p(x_2 | y) p(y)$$



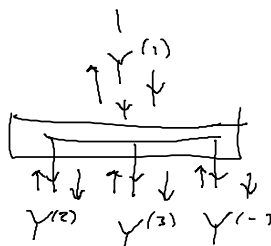


QUERY: $p(y|x_2=\bar{x}_2)$

$$p(y, x_2=\bar{x}_2) = \sum_{x_1} p(y, x_1, \bar{x}_2) = \sum_{x_1} \underbrace{p(x_1|y)}_{b_{Y(1)}(y)} \underbrace{p(\bar{x}_2|y)}_{b_{Y(2)}(y)} \underbrace{p(y)}_{f_{Y(1)}(y)}$$

$$b_{Y(2)}(y) = \sum_{x_1} p(x_1|y) u(x_1) = u(y)$$

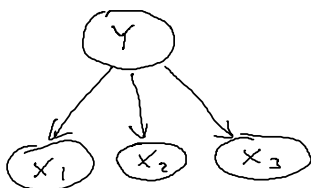
$$b_{Y(3)}(y) = b_{Y(1)}(y) b_{Y(2)}(y)$$



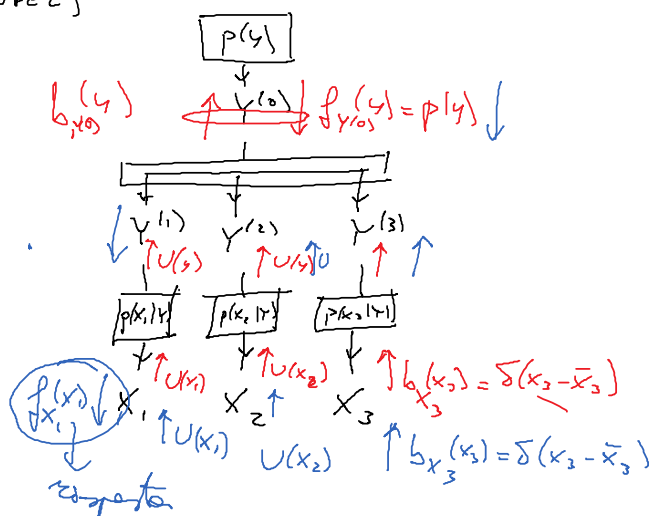
REGOLA GENERALE:

messaggio in uscita & problema dei messaggi in ingresso

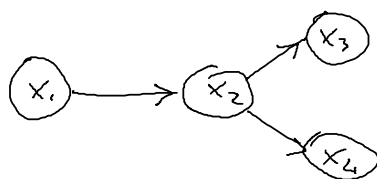
LVM (LATENT VARIABLE MODEL)



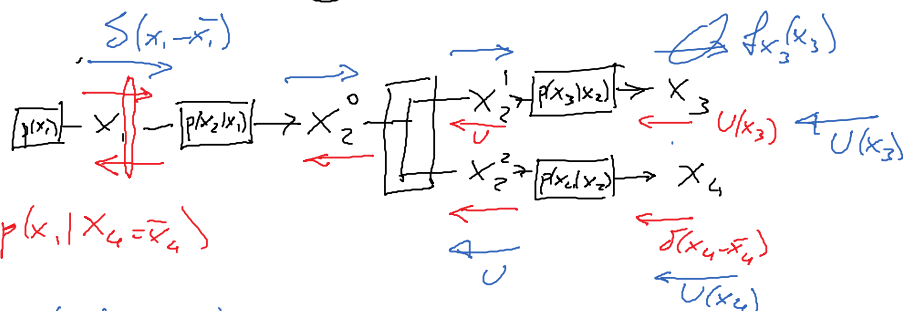
QUERY: $p(y|x_3=\bar{x}_3)$



QUERY: $p(x_1|x_3=\bar{x}_3)$



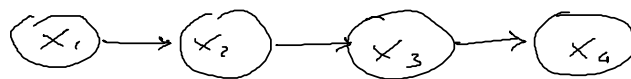
$$p(x_1, x_2, x_3, x_4) = p(x_1) p(x_2|x_1) p(x_3|x_2) p(x_4|x_2)$$



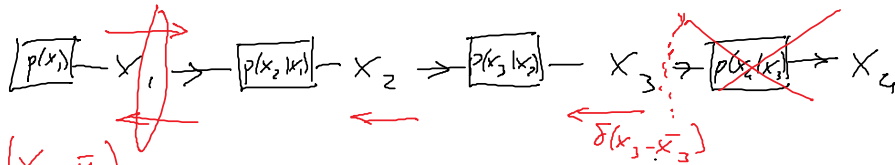
QUERY: $p(x_1|x_4=\bar{x}_4)$

QUERY: $p(x_3|x_1=\bar{x}_1)$

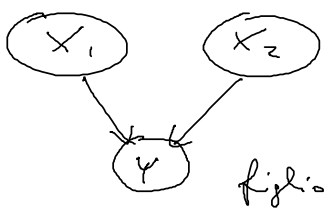
query: $p(x_3 | x_1 = \bar{x}_1)$



$$p(x_1, x_2, x_3, x_4) = p(x_1) p(x_2 | x_1) p(x_3 | x_2) p(x_4 | x_3)$$



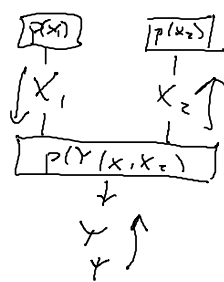
QUERY: $p(x_1 | x_3 = \bar{x}_3)$

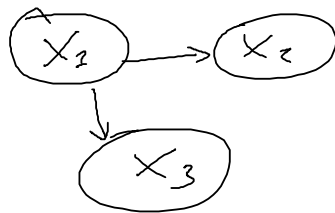


genitori

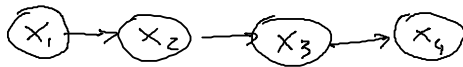
figlio

$$p(y, x_1, x_2) = p(y | x_1, x_2) p(x_1) p(x_2)$$

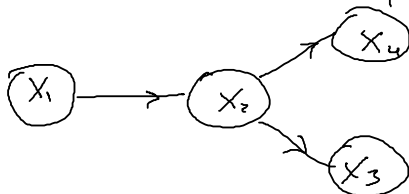


RETI BAYESIANE

$$p(x_1, x_2, x_3) = p(x_1) p(x_2 | x_1) p(x_3 | x_1)$$

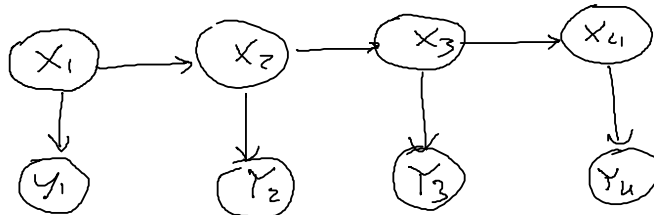


$$p(x_1, x_2, x_3, x_4) = p(x_4 | x_3) p(x_3 | x_2) p(x_2 | x_1) p(x_1)$$

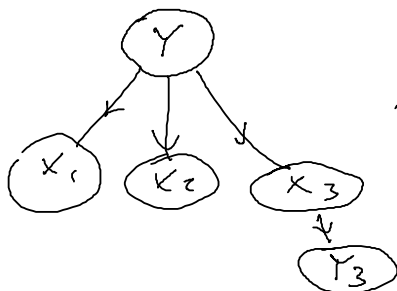


$$p(x_1, x_2, x_3, x_4) = p(x_4 | x_2) p(x_3 | x_2) p(x_2 | x_1) p(x_1)$$

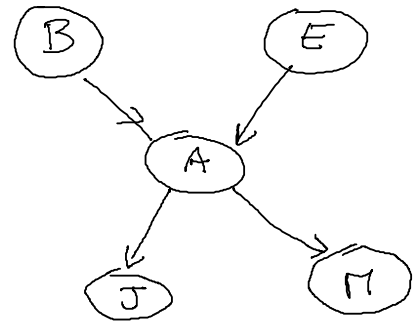
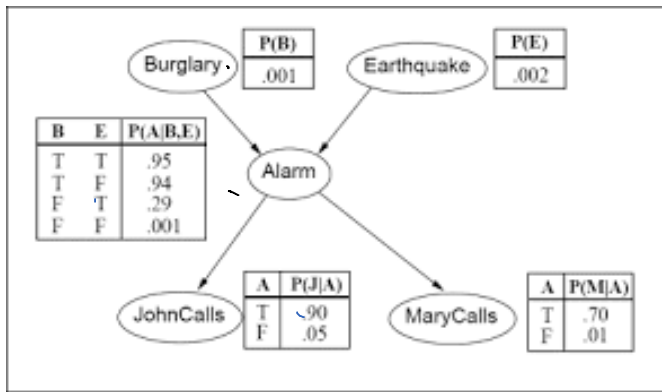
HMM
Hidden Markov
model



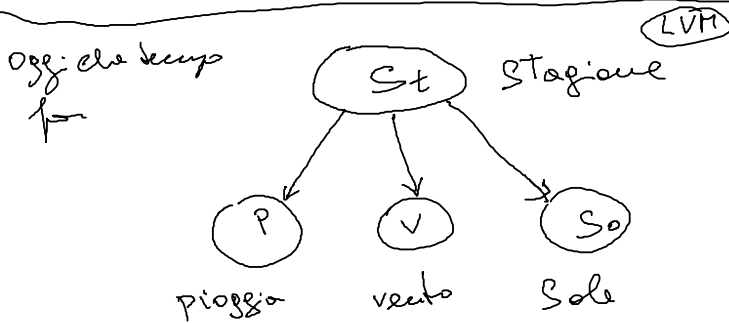
$$p(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = p(y_1 | x_1) p(x_2 | x_1) p(y_2 | x_2) p(x_3 | x_2) p(y_3 | x_3) p(x_4 | x_3) p(y_4 | x_4) p(x_1)$$



$$p(x_1, x_2, x_3, y_3, Y) = p(Y) p(x_1 | Y) p(x_2 | Y) p(x_3 | Y) p(y_3 | x_3)$$



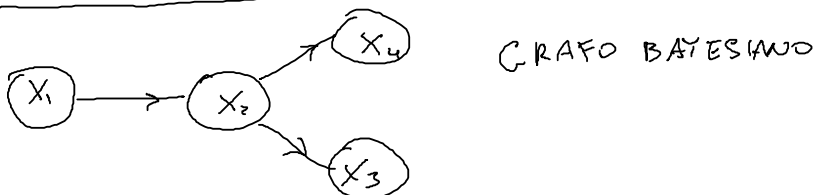
$$P(BEAJM) = P(B)P(E)P(A|BE)P(J|A)P(M|A)$$



$$P(S_t P V S_o) = P(S_t)P(P|S_t)P(V|S_t)P(S_o|S_t)$$

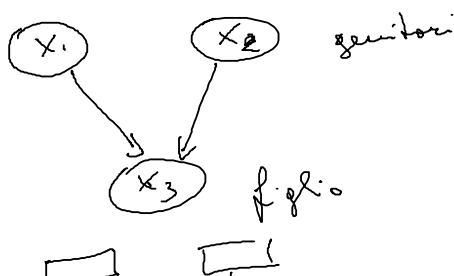
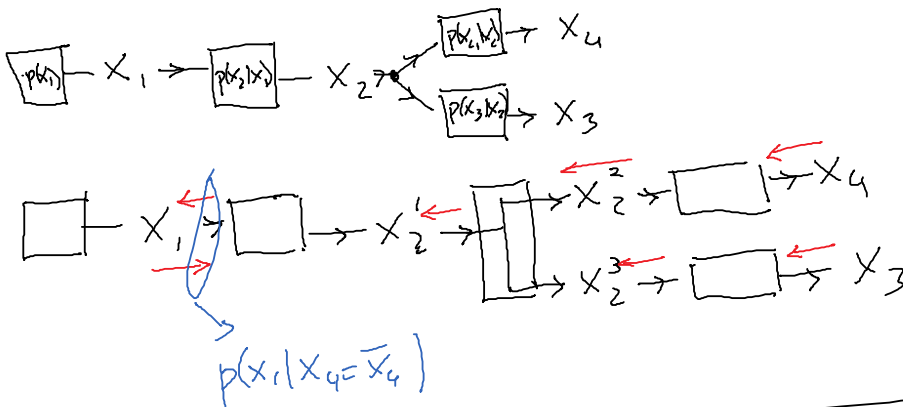
$P(S_o|S_t) =$

	Sole	
	T	F
Inverno	0.5	0.5
Estate	0.7	0.3
Autunno	0.8	0.2
Primavera	0.6	0.4



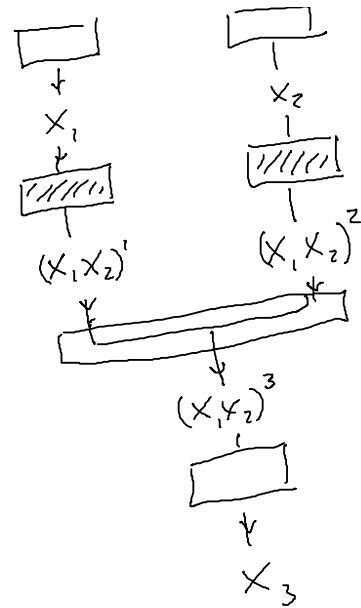
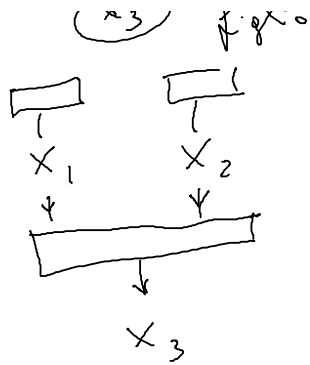
$$P(X_1, X_2, X_3, X_4) = P(X_4|X_2)P(X_3|X_2)P(X_2|X_1)P(X_1)$$

GRAFO FATTORIALE



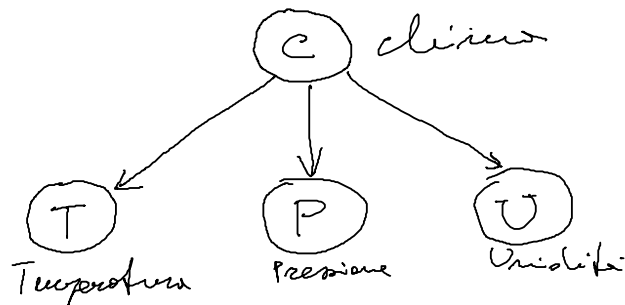
$$P(X_1, X_2, X_3) = P(X_3|X_1, X_2)P(X_1)P(X_2)$$





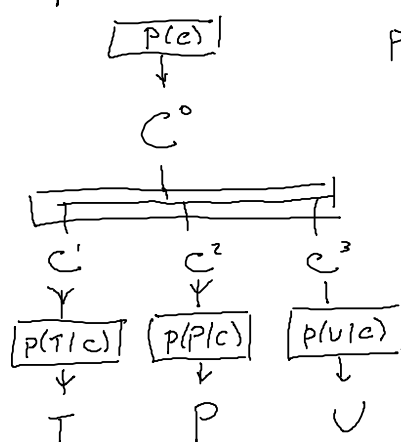
Es. FUSIONE DA SENSORI

$C \in \{\text{Bello, Brutto}\}$
 $\{b, B\}$



$T \in \{b, a\}$ $P \in \{b, a\}$ $U \in \{b, a\}$

$$p(c, T, P, U) = p(c) p(T|c) p(P|c) p(U|c)$$



$$p(c) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$p(T|c) = \begin{array}{c|cc} c \backslash T & b & a \\ \hline b & 0.3 & 0.7 \\ B & 0.6 & 0.4 \end{array}$$

$$p(P|c) = \begin{array}{c|cc} c \backslash P & b & a \\ \hline b & 0.2 & 0.8 \\ B & 0.5 & 0.5 \end{array}$$

$$p(U|c) = \begin{array}{c|cc} c \backslash U & b & a \\ \hline b & 0.4 & 0.6 \\ B & 0.7 & 0.3 \end{array}$$

QUERY: Dato che $T=a$, calcolare la dist. di U $p(U|T=a)$

$$b_{c1} \propto p(T|c) b_T$$

$$= \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$$

$$f_{c0} \propto p(c) f_{c0}$$

$$f_{c0} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$p \propto p_{c1} \odot b_{c2} \propto b_{c1} = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$$

$$b_{c^2} \propto \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$b_{c^3} \propto p_{c^0} \odot b_{c^1} \odot b_{c^2} \propto b_{c^1} = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$$

$$f_v \propto P(v|e)^T f_{c^3} = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.56 \\ 0.54 \end{pmatrix}$$

$$p_v \propto f_v \odot b_v \propto \begin{pmatrix} 0.56 \\ 0.54 \end{pmatrix}$$

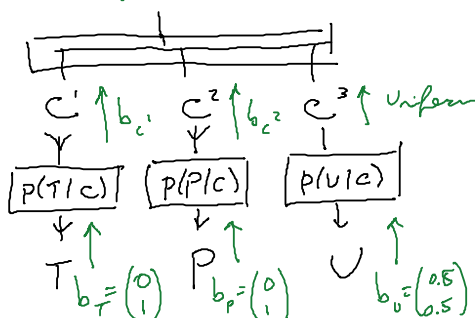
Normalise $\rightarrow p_v = \begin{pmatrix} 0.5091 \\ 0.4909 \end{pmatrix}$

QUERY: $T=a, P=a$

$$p(c|T=a, P=a)?$$

$$b_{c^1} \propto P(T|c) b_T = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$$

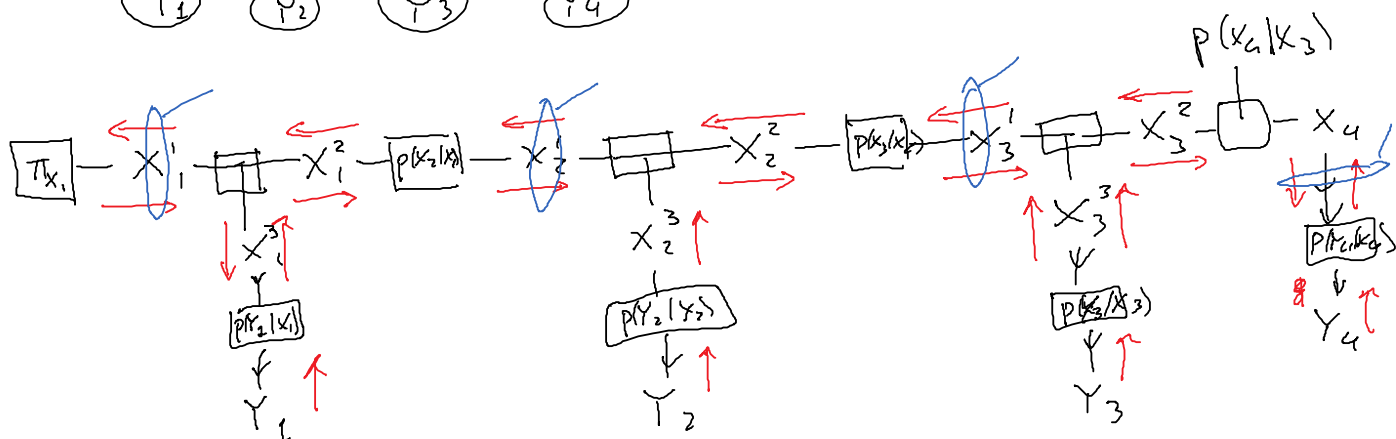
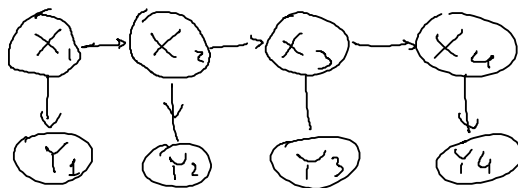
$$b_{c^2} \propto P(P|c) b_P = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.5 \end{pmatrix}$$



$$p(c|T=a, P=a) = p_c \propto \begin{pmatrix} 0.56 \\ 0.20 \end{pmatrix} \Rightarrow \text{Normalise}$$

$$p_c = \begin{pmatrix} 0.7368 \\ 0.2632 \end{pmatrix}$$

HMM



$$? p(x_i | Y_1 = \bar{y}_1, Y_2 = \bar{y}_2, Y_3 = \bar{y}_3, Y_4 = \bar{y}_4) \quad i=1,2,3,4$$