THE GAUSSIAN MULTIVARIATE

CLASS NOTES

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$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \quad \text{Random vector} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$$

$$E[\underline{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{bmatrix} = \underline{\mu} \quad \text{Mean}$$

$$E[\underline{X}\,\underline{X}^T] = \underline{\underline{R}}_x \quad \text{AUTOCORRELATION MATRIX}$$

$$E[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^T] = \underline{\underline{\Sigma}}_x \quad \text{COVARIANCE MATRIX}$$

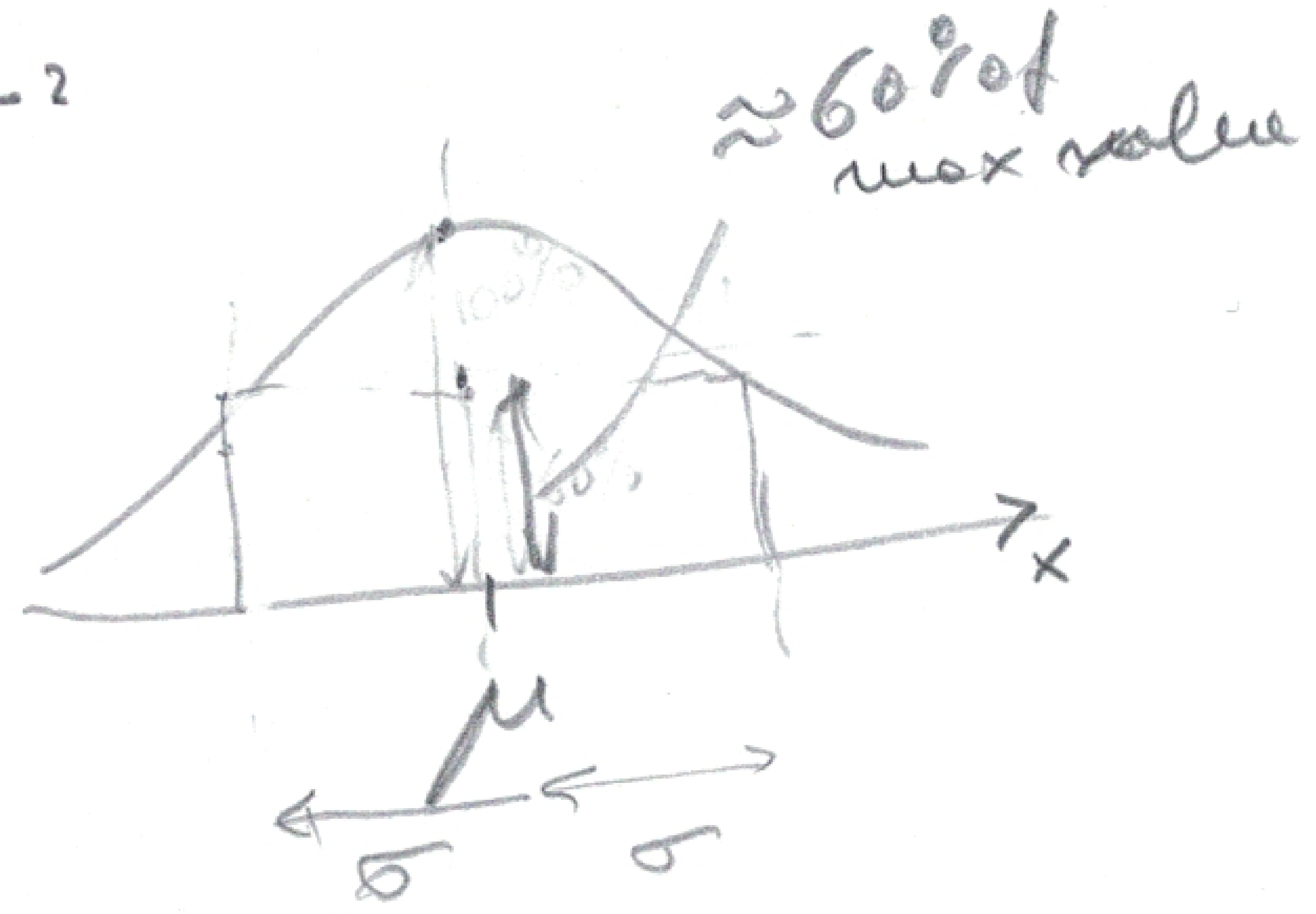
GAUSSIAN PDF

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{N/2}\,|\underline{\underline{\Sigma}}_x|^{1/2}}\, e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{\underline{\Sigma}}^{-1}(\underline{x}-\underline{\mu})}$$

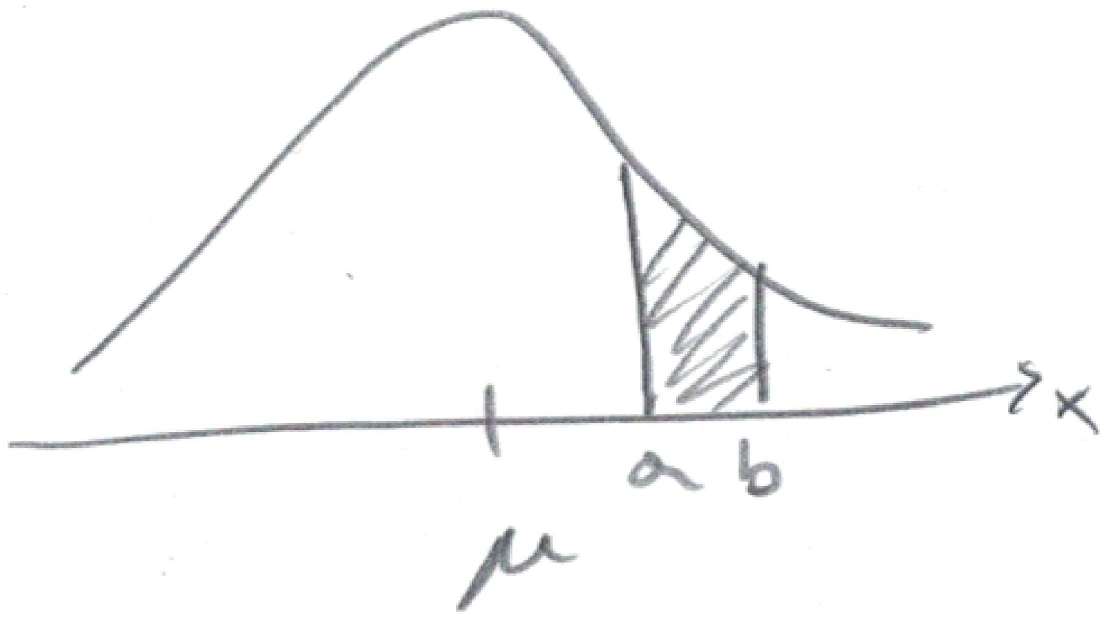
ONE-DIMENSIONAL GAUSSIAN

Clearly for $\underline{N=1}$, $\underline{\mu} = \mu$, $\underline{\underline{\Sigma}} = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$



To integrate the Gaussian pdf



$$\int_a^b N(x;\mu,\sigma^2)\,dx = \Pr\{x\in[a,b]\}$$

We use the standardised Gaussian $\mu=0,\ \sigma=1$

$$N(x;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$Q(x)=\int_x^\infty N(\xi;0,1)\,d\xi$

$\phi(x)$

$" \int_{-\infty}^{x} N(\xi;0,1)\,d\xi$

$$\phi(x)=1-Q(x)$$
$$Q(-x)=1-Q(x)$$

$$\Pr\{a\le x\le b\}=\phi\left(\frac{b-\mu}{\sigma}\right)-\phi\left(\frac{a-\mu}{\sigma}\right)=Q\left(\frac{a-\mu}{\sigma}\right)-Q\left(\frac{b-\mu}{\sigma}\right)$$

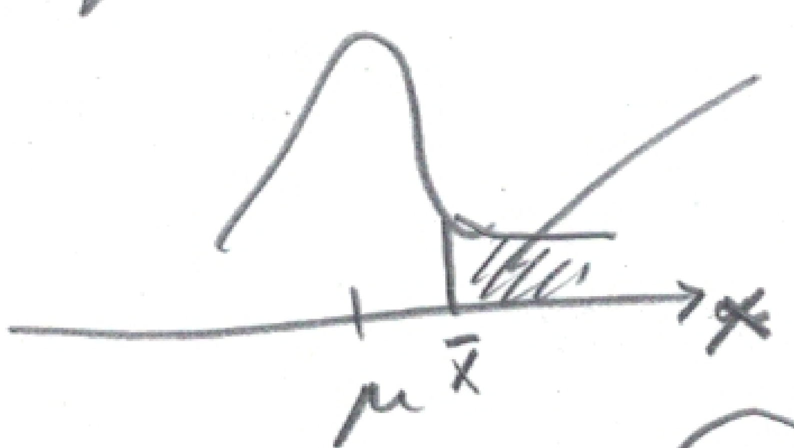
Proof:

$$\int_a^b \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}\cancel{\sigma}}\cancel{\sigma}\,e^{-\frac{\xi^2}{2}}d\xi=\phi\left(\frac{b-\mu}{\sigma}\right)-\phi\left(\frac{a-\mu}{\sigma}\right)$$

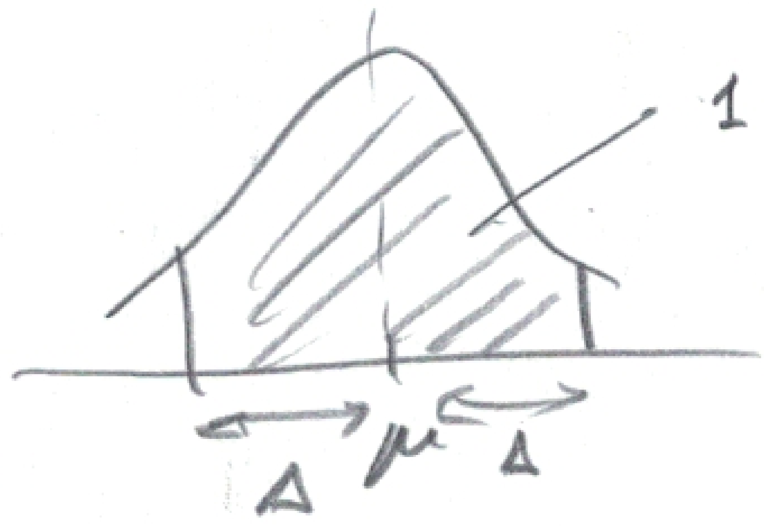
$$\xi=\frac{x-\mu}{\sigma}$$

$$d\xi=\frac{1}{\sigma}dx$$

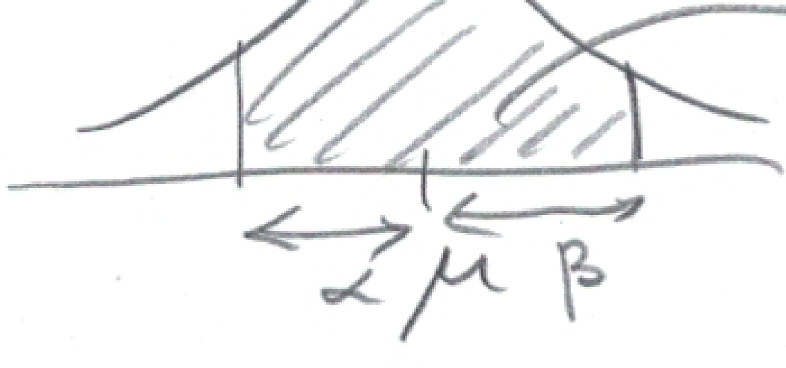
Examples



$Q\left(\frac{\bar{x}-\mu}{\sigma}\right)$



$1-2Q\left(\frac{\Delta}{\sigma}\right)$



$1-Q\left(\frac{\beta}{\sigma}\right)-Q\left(\frac{\alpha}{\sigma}\right)$

The two-dimensional case deserves some attention

$N=2$ Name $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$ $\quad E[X] \triangleq \bar{x}$

$E[Y] \triangleq \bar{y}$

$$\mathrm{Cov}(X,Y) = E[(X-\bar{x})(Y-\bar{y})] \triangleq \sigma_{xy}$$

$$\underline{\underline{\Sigma}} = \begin{bmatrix} E[(X-\bar{x})^2] & E[(X-\bar{x})(Y-\bar{y})] \\ E[(X-\bar{x})(Y-\bar{y})] & E[(Y-\bar{y})^2] \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

CORRELATION COEFFICIENT

$$\rho \triangleq \frac{E[(X-\bar{x})(Y-\bar{y})]}{\sigma_x \sigma_y} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \qquad \left\{ \begin{array}{l} \sigma_x = \sqrt{\sigma_x^2} \\ \sigma_y = \sqrt{\sigma_y^2} \end{array} \right.$$

$$\sigma_{xy} = \sigma_x \sigma_y \rho$$

$$0 \leq \rho \leq 1$$

The inverse of a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|\Sigma| = \sigma_x^2 \sigma_y^2 - \sigma_{xy}^2$$

$$\underline{\underline{\Sigma}}^{-1} = \frac{1}{|\underline{\underline{\Sigma}}|} \begin{bmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{bmatrix}$$

$$= \frac{1}{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2} \begin{bmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{bmatrix} =$$

$$f_{XY}(x,y) = \frac{1}{2\pi(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)^{1/2}} e^{-\frac{1}{2} \frac{[X-\bar{x} \;\; Y-\bar{y}] \begin{bmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{bmatrix} \begin{bmatrix} X-\bar{x} \\ Y-\bar{y} \end{bmatrix}}{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2}}$$

$$[(X-\bar{x})\sigma_y^2 - \sigma_{xy}(y-\bar{y}) \quad -\sigma_{xy}(x-\bar{x}) + \sigma_x^2(y-\bar{y})] \begin{bmatrix} X-\bar{x} \\ y-\bar{y} \end{bmatrix}$$

$$= (X-\bar{x})^2 \sigma_y^2 - \sigma_{xy}(y-\bar{y})(x-\bar{x}) - \sigma_{xy}(x-\bar{x})(y-\bar{y}) + \sigma_x^2 (y-\bar{y})^2$$

$$= (X-\bar{x})^2 \sigma_y^2 - 2\sigma_{xy}(y-\bar{y})(x-\bar{x}) + \sigma_x^2 (y-\bar{y})^2$$

$$\frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}}\cdot$$

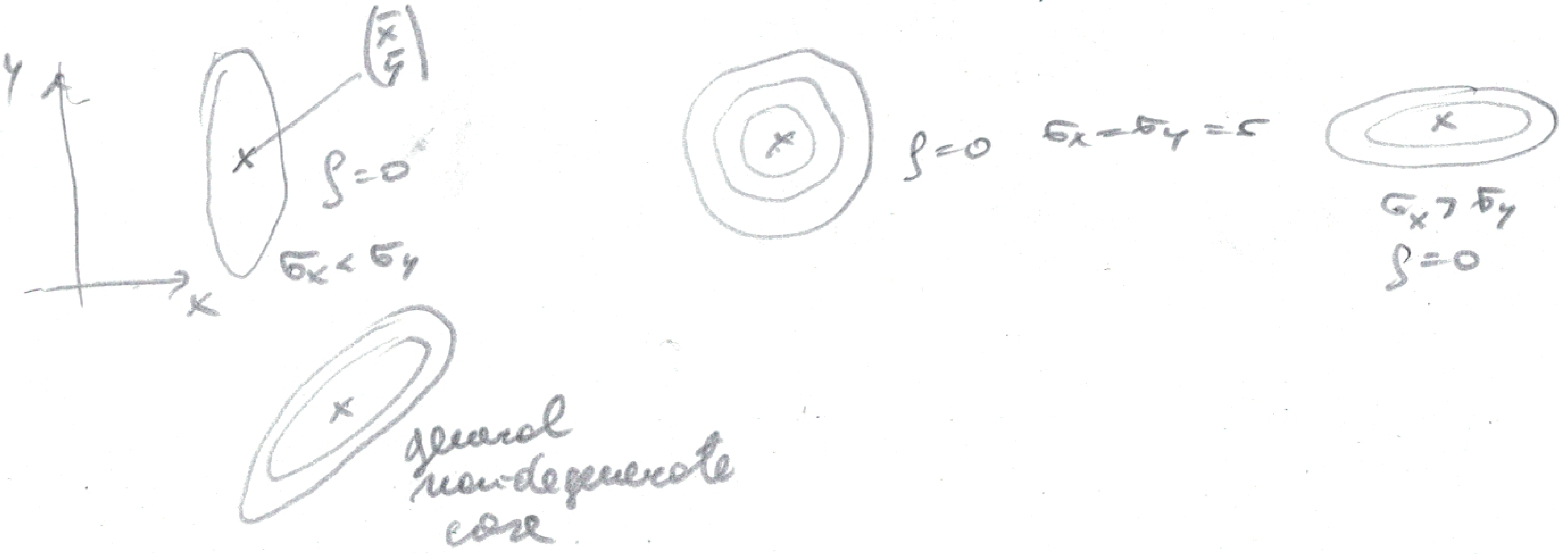
$$\exp\left(-\frac{1}{2}\,\frac{(x-\bar{x})^2\sigma_y^2-2\sigma_{xy}(x-\bar{x})(y-\bar{y})+\sigma_x^2(y-\bar{y})^2}{\sigma_x^2\sigma_y^2-\sigma_{xy}^2}\right)$$

$$=\frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}}$$

$$\exp\left(-\frac{1}{2}\,\frac{\sigma_y^2(x-\bar{x})-2\sigma_{xy}(x-\bar{x})(y-\bar{y})+\sigma_x^2(y-\bar{y})^2}{\sigma_x^2\sigma_y^2(1-\rho^2)}\right)$$

$$=\frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}}\exp\left[-\frac{\frac{x-\bar{x}}{\sigma_x^2}-2\frac{\sigma_{xy}}{\sigma_x^2\sigma_y^2}(x-\bar{x})(y-\bar{y})+\frac{1}{\sigma_y^2}(y-\bar{y})^2}{2(1-\rho^2)}\right]$$

$$\boxed{f_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}}\,e^{-\frac{\frac{(x-\bar{x})^2}{\sigma_x^2}-\frac{2\rho}{\sigma_x\sigma_y}(x-\bar{x})(y-\bar{y})+\frac{(y-\bar{y})^2}{\sigma_y^2}}{2(1-\rho^2)}}}$$



The conditional density $f_{Y|X}(y|x)$ reveals better the relation between x and y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}(x-\bar{x})^2}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\cancel{\sqrt{2\pi}\,\sigma_x}}{\sqrt{2\pi}\,\cancel{\sigma_x}\sigma_y(1-\rho^2)^{1/2}} \cdot$$

$$\cdot \exp\left[-\frac{\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2r(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2}}{2(1-\rho^2)} + \frac{(x-\bar{x})^2}{2\sigma_x^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_y(1-\rho^2)^{1/2}} \exp\left(-\frac{\frac{(x-\bar{x})^2}{\cancel{\sigma_x^2}}\cancel{\sigma_x^2} - \frac{2r(x-\bar{x})(y-\bar{y})}{\cancel{\sigma_x}\sigma_y}\sigma_x^{\cancel{2}} + \frac{(y-\bar{y})^2}{\sigma_y^2}\sigma_x^2 - (1-\rho^2)(x-\bar{x})^2}{2\sigma_x^2(1-\rho^2)}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{\rho^2(x-\bar{x})^2 - 2r(x-\bar{x})(y-\bar{y})\frac{\sigma_x}{\sigma_y} + (y-\bar{y})^2\frac{\sigma_x^2}{\sigma_y^2}}{2\sigma_x^2(1-\rho^2)}\right]$$

$$\boxed{f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_y(1-\rho^2)^{1/2}} \exp - \frac{\left((y-\bar{y})\frac{\sigma_x}{\sigma_y} - \rho(x-\bar{x})\right)^2}{2\sigma_x^2(1-\rho^2)}}$$

Expanding the numerator of the exponent

$$\left(\frac{\sigma_x}{\sigma_y}y - \frac{\sigma_x}{\sigma_y}\bar{y} - \rho(x-\bar{x})\right)^2 = \frac{\sigma_x^2}{\sigma_y^2}\left(y-\bar{y}-\rho\frac{\sigma_y}{\sigma_x}(x-\bar{x})\right)^2$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{\left[y-\left(\bar{y}+\rho\frac{\sigma_y}{\sigma_x}(x-\bar{x})\right)\right]^2}{2\sigma_y^2(1-\rho^2)}\right]$$

The conditional mean (best MSE estimator of y given x)

$$E[Y|x] = \bar{y} + \rho\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

linear estimator (affine)

(This is consistent with the general result that will be presented later)

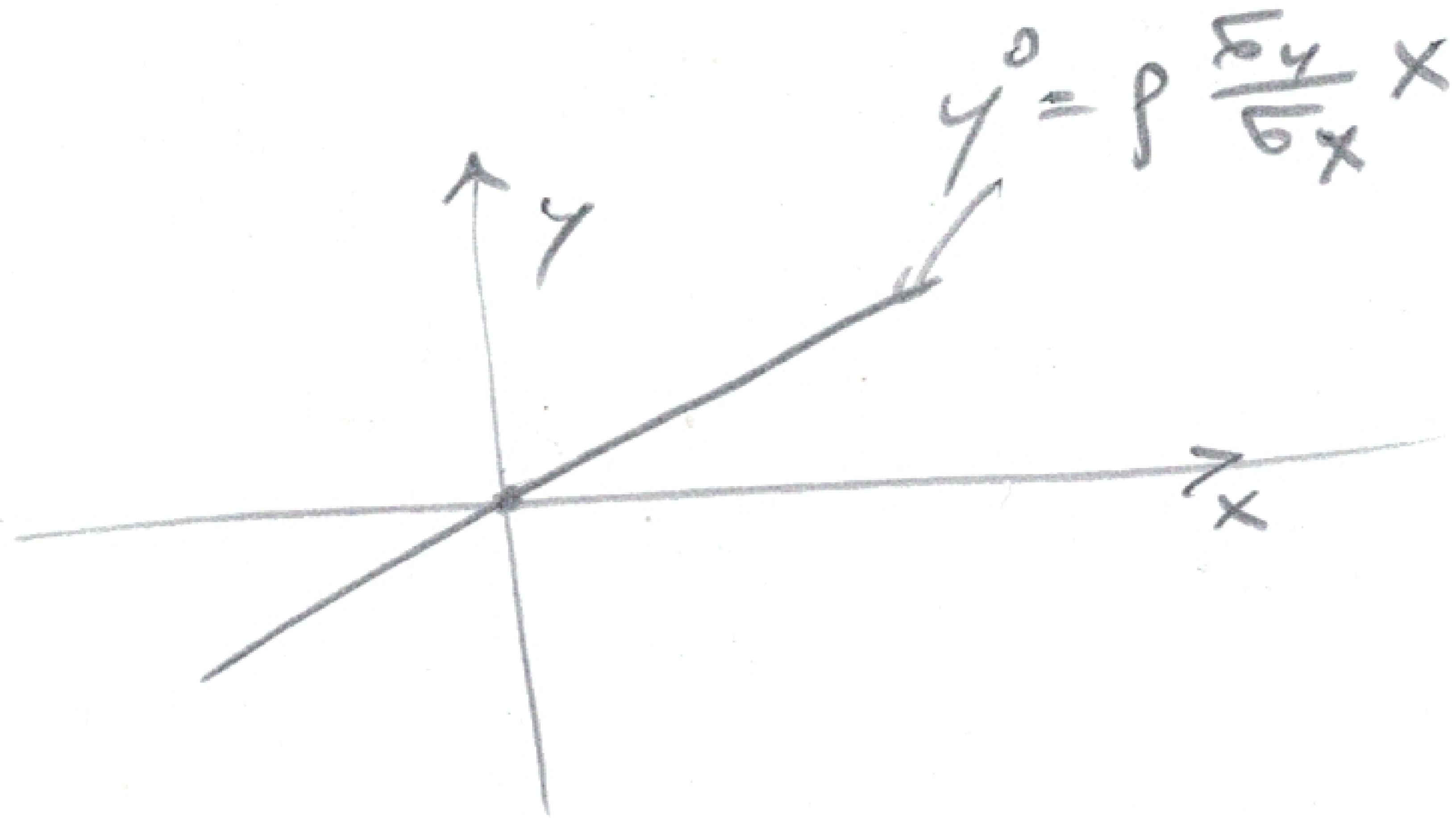
$$\boxed{y^{OPT} = \bar{y} + \rho \frac{\sigma_y}{\sigma_x}(x-\bar{x})}$$

The estimator is unbiased because

$$E[y^{OPT}] = \bar{y} + \rho \frac{\sigma_y}{\sigma_x} \underbrace{E[x-\bar{x}]}_{0} = \bar{y}$$

Looking at the de-meaned problem

$\bar{y}=0 \quad \bar{x}=0$, or equivalently shifting the origin on $\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}$.



$$y = \frac{\sigma_{xy}}{\sigma_x \cancel{\sigma_y}} \frac{\cancel{\sigma_y}}{\sigma_x} x = \frac{\sigma_{xy}}{\sigma_x^2} x$$

TWO ~~M~~ MULTI-DIMENSIONAL RANDOM VARIABLES JOINTLY GAUSSIAN

Consider the two vector-random variables

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_L \end{bmatrix}, \qquad (A9)$$

with dimensions N and L respectively.

A compact representation of their marginal and joint statistics is expressed considering the random variable

$$\underline{\underline{Z}} = \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \\ Y_1 \\ \vdots \\ Y_L \end{bmatrix} \qquad (A10)$$

with size $M = L + N$. Define the means and the covariance matrices of $\underline{X}$ and $\underline{Y}$ respectively as:

$$E[\underline{X}] = \underline{\bar{x}} \quad , \quad E[\underline{Y}] = \underline{\bar{y}} \qquad (A11)$$

$$\mathrm{Cov}[\underline{X}] = \underline{\underline{\Sigma}}_{\underline{X}} \qquad \mathrm{cov}[\underline{Y}] = \underline{\underline{\Sigma}}_{Y} \,. \qquad (A12)$$

the cross-covariance between $\underline{X}$ and $\underline{Y}$ is

$$\mathrm{Cov}[\underline{X}\,\underline{Y}] = E\left[(\underline{X} - \underline{\bar{x}})(\underline{Y} - \underline{\bar{y}})^T\right] = \Sigma_{\underline{XY}} \,. \qquad (A13)$$

Note that this matrix is not necessarily symmetric.

but $\Sigma_{XY}^T = \Sigma_{YX}$.

Now the covariance of $\underline{Z}$ is

$$\mathrm{cov}[\underline{z}] = E\left[(\underline{Z}-\bar{\underline{z}})(\underline{z}-\bar{\underline{z}})^T\right]$$

$$= E\left[\left(\begin{bmatrix}(\underline{X} - \bar{\underline{x}})\\(\underline{Y} - \bar{\underline{y}})\end{bmatrix}\right)\left(\underline{X}^T-\bar{\underline{x}}^T \mid \underline{Y}^T-\bar{\underline{y}}^T\right)\right]$$

$$= \begin{bmatrix}\Sigma_{\underline{X}} & \Sigma_{\underline{X}\underline{Y}}\\ \Sigma_{\underline{Y}\underline{X}} & \Sigma_{\underline{Y}}\end{bmatrix} \triangleq \underline{\underline{\Sigma}}_z \qquad \text{(A14)}$$

The density of $\underline{Z}$, assumed to be gaussian is written as in (A2)

To evaluate the conditional density $P_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y})$ we can write

$$P_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = \frac{P_{\underline{X}\underline{Y}}(\underline{x},\underline{y})}{P_{\underline{Y}}(\underline{y})} = \frac{P_{\underline{z}}(\underline{z})}{P_{\underline{Y}}(\underline{y})} \qquad \text{(A15)}$$

Knowing that $\underline{Y}$ is also gaussian we can write

$$P_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = \frac{(2\pi)^{L/2}|\Sigma_y|^{1/2}}{(2\pi)^{M/2}|\Sigma_{\underline{z}}|^{1/2}} \cdot$$

$$\cdot \frac{\exp\left\{-\frac{1}{2}\left[(\underline{x}^T-\bar{\underline{x}}^T) \mid \underline{y}^T-\bar{\underline{y}}^T\right]^T \Sigma_z^{-1}\begin{bmatrix}\underline{x}-\bar{\underline{x}}\\ \underline{y}-\bar{\underline{y}}\end{bmatrix}\right\}}{\exp\left\{-\frac{1}{2}(\underline{y}^T-\bar{\underline{y}}^T)\Sigma_y^{-1}(\underline{y}-\bar{\underline{y}})\right\}} \qquad \text{(A16)}$$

let us use the following formula

$$\begin{bmatrix} I & -\Sigma_{xy}\Sigma_y^{-1} \\ 0 & I \end{bmatrix} \Sigma_z \begin{bmatrix} I & 0 \\ -\Sigma_y^{-1}\Sigma_{xy}^{T} & I \end{bmatrix} = \begin{bmatrix} \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx} & 0 \\ 0 & \Sigma_y \end{bmatrix} \quad \text{(A17)}$$

It can be checked with direct substitution. (Anderson and Moore, 1979, p. 25).

Taking the determinant of (A17) we get:

$$|\Sigma_z| = |\Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}|\,|\Sigma_y|. \quad \text{(A18)}$$

Taking the inverse of both sides of (A17) we get

$$\begin{bmatrix} I & 0 \\ -\Sigma_y^{-1}\Sigma_{xy} & I \end{bmatrix}^{-1} \Sigma_z^{-1} \begin{bmatrix} I & -\Sigma_{xy}\Sigma_y^{-1} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} (\Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx})^{-1} & 0 \\ 0 & \Sigma_y^{-1} \end{bmatrix} \quad \text{(A19)}$$

then

$$\Sigma_z^{-1} = \begin{pmatrix} I & 0 \\ -\Sigma_y^{-1}\Sigma_{xy} & I \end{pmatrix} \begin{bmatrix} (\Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx})^{-1} & 0 \\ 0 & \Sigma_y^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{xy}\Sigma_y^{-1} \\ 0 & I \end{bmatrix} \quad \text{(A20)}$$

Now the first exponent in (A16) becomes

$$(\underline{x}^T-\bar{\underline{x}} \mid \underline{y}^T-\bar{\underline{y}}^T)\Sigma_z^{-1}(\underline{x}^T-\bar{\underline{x}}^T \mid \underline{y}^T-\bar{\underline{y}}^T)^T$$

$$=(\underline{x}^T-\bar{\bar{\underline{x}}}^T)(\Sigma_x-\Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy})^{-1}(\underline{x}-\bar{\bar{\underline{x}}})+(\underline{y}^T-\bar{\underline{y}}^T)\Sigma_y^{-1}(\underline{y}-\bar{\underline{y}}) \qquad (A21)$$

with

$$\bar{\bar{\underline{x}}}=\bar{\underline{x}}+\Sigma_{xy}\Sigma_y^{-1}(\underline{y}-\bar{\underline{y}}) \,. \qquad (A22)$$

therefore

$$P_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y})=\frac{1}{(2\pi)^{N/2}\left|\Sigma_x-\Sigma_{xY}\Sigma_y^{-1}\Sigma_{yx}\right|^{1/2}}\,\cdot \qquad (A23)$$

$$\cdot\exp\left[-\frac{1}{2}(\underline{x}^T-\bar{\bar{\underline{x}}}^T)(\Sigma_x-\Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx})^{-1}(\underline{x}-\bar{\bar{\underline{x}}})\right]$$

the conditional mean of X given Y=y is then

$$E[X|Y=y]=\bar{\underline{x}}+\Sigma_{XY}\Sigma_Y^{-1}(\underline{y}-\bar{\underline{y}}) \qquad (A24)$$

the conditional covariance is

$$\Sigma_x-\Sigma_{XY}\Sigma_y^{-1}\Sigma_{yx} \,. \qquad (A25)$$