Analog Communications

— Lecture 02 —
Baseband Representation of Signals

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Outline

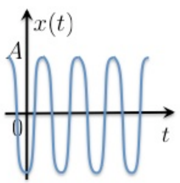
Phase Vector

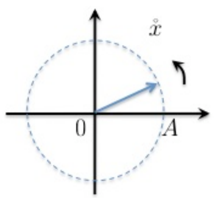
In engineering, a sinusoidal signal

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$

is commonly described through a phasor

$$\mathring{x}(t) = A\exp(j(2\pi f_0 t + \varphi))$$

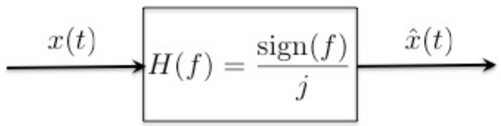


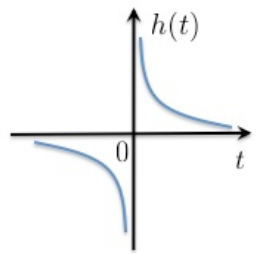


It is useful to replace linear differential equations with algebraic equations

Hilbert Transform (1/2)

The Hilbert transform of $x(t) \in R$ is the real-valued signal obtained as





$$\begin{aligned}\hat{x}(t) &= \mathcal{H}\{x(t)\} \triangleq x(t) \star \frac{1}{\pi t}\\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{x(\tau)}{t-\tau}d\tau\end{aligned}$$

$$\mathcal{F}\{\hat{x}(t)\}(f) = X(f)\mathcal{F}\left\{\frac{1}{\pi t}\right\}(f) = X(f)\frac{\text{sign}(f)}{j}$$

Hilbert Transform (2/2)

The Hilbert inverse transform is simply obtained applying it twice

$$\hat{\hat{x}}(t) = \mathcal{H}\{\hat{x}(t)\} = x(t) \star \frac{1}{\pi t} \star \frac{1}{\pi t}$$

In Fourier domain

$$\mathcal{F}\{\mathcal{H}\{\hat{x}(t)\}\}(f) = X(f)\frac{\mathrm{sign}(f)}{j}\frac{\mathrm{sign}(f)}{j} = -X(f)$$

Back to time domain

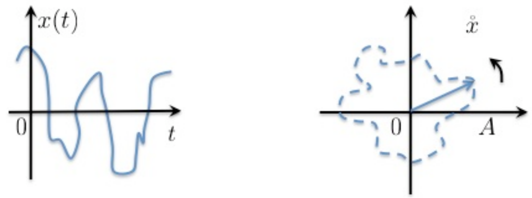
$$\mathcal{H}\{\hat{x}(t)\} = -x(t)$$

Equations for Hilbert (direct and inverse) transforms are

$$\hat{x}(t) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x(\tau)}{t-\tau}d\tau \qquad x(t) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\hat{x}(\tau)}{t-\tau}d\tau$$

Analytic Signal (1/3)

The analytic signal is a generalization of the concept of phasor to describe an arbitrary real-valued signal



The following relation must hold as well

$$x(t) = \Re\{\mathring{x}(t)\}$$

Analytic Signal (2/3)

Consider $x(t) \in \mathbb{R}$ and its Fourier transform

$$X(f) = \mathcal{F}\{x(t)\}(f) = \int_{\mathbb{R}} x(t)\exp(-j2\pi f t)dt$$

known to be Hermitian, i.e.

$$X(f) = X^*(-f)$$

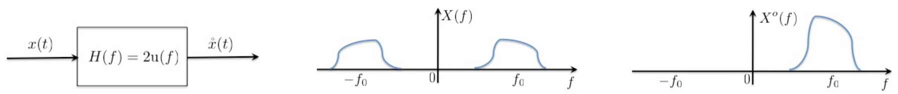
or else

$$\begin{cases} |X(-f)| = |X(f)| \\ \angle X(-f) = -\angle X(f) \end{cases} \qquad \begin{cases} \Re\{X(-f)\} = \Re\{X(f)\} \\ \Im\{X(-f)\} = -\Im\{X(f)\} \end{cases}$$

Knowing the behavior only for positive frequencies is enough

Analytic Signal (3/3)

$x(t)$ and $\mathring{x}(t)$, viewed as I/O of the following LTI system, represent the same information



Also, from $\mathring{X}(f) = 2\mathrm{u}(f)X(f)$ we get

$$\begin{aligned} \mathring{x}(t) &= \mathcal{F}^{-1}\{2\mathrm{u}(f)\} \star x(t) = \mathcal{F}^{-1}\{1 + \mathrm{sign}(f)\} \star x(t) \\ &= \left(\delta(t) + j\frac{1}{\pi t}\right) \star x(t) = x(t) + j\frac{1}{\pi t} \star x(t) \\ &= x(t) + j\hat{x}(t) \\ &= \rho(t)\exp(\phi(t)) \end{aligned}$$

$\mathring{x}(t)$ is the analytic signal representing $x(t)$

Properties

$$x(t)=\sum_n a_n x_n(t) \qquad \Rightarrow \qquad \begin{cases} \hat{x}(t)=\sum_n a_n \hat{x}_n(t) \\ \mathring{x}(t)=\sum_n a_n \mathring{x}_n(t) \end{cases}$$

$$\begin{cases} z(t)=x(t)y(t) \\ x(t)\ \text{low}-\text{pass in}\ (0,B_1) \\ y(t)\ \text{band}-\text{pass in}\ (B_2,B_3) \\ B_1<B_2<B_3 \end{cases} \qquad \Rightarrow \qquad \mathring{z}(t)=x(t)\mathring{y}(t)$$

$$z(t)=x(t)\star y(t) \qquad \Rightarrow \qquad \begin{aligned} \mathring{z}(t) &= \frac{1}{2}\mathring{x}(t)\star\mathring{y}(t) \\ &= \mathring{x}(t)\star y(t) \\ &= x(t)\star\mathring{y}(t) \end{aligned}$$

Correlation Properties

Using the standard I/O correlation relationships for LTI systems

$$\begin{aligned} y(t) &= h(t)\star x(t) \\ R_{yy}(\tau) &= h(\tau)\star h^*(-\tau)\star R_{xx}(\tau) \end{aligned}$$

we get

$$\begin{aligned} R_{\hat{x}\hat{y}}(\tau) &= R_{xy}(\tau) \\ R_{\mathring{x}\mathring{y}}(\tau) &= 2\mathring{R}_{xy}(\tau) \\ &= 2R_{xy}(\tau)+j2\hat{R}_{xy}(\tau) \\ R_{xy}(\tau) &= \frac{1}{2}\Re\{R_{\mathring{x}\mathring{y}}(\tau)\} \end{aligned}$$

PSD Properties (1/2)

$$P_x=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}|x(t)|^2dt=\int_{\mathbb{R}}\underbrace{\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2}_{P_x(f)}df$$

$$\begin{aligned} P_{\hat{x}} &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}|\hat{x}(t)|^2dt \\ &= \int_{\mathbb{R}}\underbrace{\lim_{T\to\infty}\frac{1}{T}\left|\frac{\text{sign}(f)}{j}\right|^2|X_T(f)|^2}_{P_{\hat{x}}(f)}df \\ &= \int_{\mathbb{R}}\underbrace{\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2}_{P_{\hat{x}}(f)=P_x(f)}df \\ &= P_x \end{aligned}$$

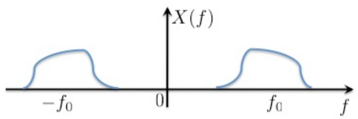
PSD Properties (2/2)

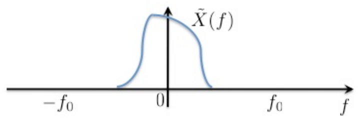
$$\begin{aligned} P_{\mathring{x}} &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}|\mathring{x}(t)|^2dt \\ &= \int_{\mathbb{R}}\underbrace{\lim_{T\to\infty}\frac{1}{T}|2\text{u}(f)|^2|X_T(f)|^2}_{P_{\mathring{x}}(f)}df \\ &= \int_{\mathbb{R}}\underbrace{4\text{u}(f)\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2}_{P_{\mathring{x}}(f)=4\text{u}(f)P_x(f)}df \\ &= 2P_x \\ &= P_x+P_{\hat{x}} \end{aligned}$$

Band-Pass Signals and Complex Envelope

A band-pass signal $x(t)$ has significant spectral components within a finite range of frequencies not including the origin, i.e. $0 < B_1 < B_2$

$$X(f) \approx 0 \qquad \forall\, |f| \in \mathbb{R} - (B_1, B_2)$$





The complex envelope (or equivalent low-pass signal) of $x(t)$ is

$$\tilde{x}(t) = \mathring{x}(t)\exp(-j2\pi f_0 t)$$

where f_0 is usually the centroid of the spectrum

$$f_0 = \frac{\int_0^\infty f P_x(f)df}{\int_0^\infty P_x(f)df}$$

I-Q Components and Rice Representation

In-Phase and Quadrature components of a band-pass signal $x(t)$ are defined as

$$\begin{aligned} x_I(t) &= \Re\{\tilde{x}(t)\} \\ x_Q(t) &= -\Im\{\tilde{x}(t)\} \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} \mathring{x}(t) &= \tilde{x}(t)\exp(+j2\pi f_0 t) \\ \tilde{x}(t) &= x_I(t) - jx_Q(t) \end{aligned}$$

and finally

$$x(t) = x_I(t)\cos(2\pi f_0 t) + x_Q(t)\sin(2\pi f_0 t)$$

Some Relations

$$\begin{aligned} \tilde{x}(t) &= \rho(t)\exp(\theta(t)) \\ x(t) &= \rho(t)\cos(2\pi f_0 t + \theta(t)) \\ \hat{x}(t) &= x_I(t)\sin(2\pi f_0 t) - x_Q(t)\cos(2\pi f_0 t) \end{aligned}$$

$$\begin{aligned} x_I(t) &= x(t)\cos(2\pi f_0 t) + \hat{x}(t)\sin(2\pi f_0 t) \\ x_Q(t) &= \hat{x}(t)\cos(2\pi f_0 t) - x(t)\sin(2\pi f_0 t) \end{aligned}$$

$$\begin{aligned} \rho(t) &= |\mathring{x}(t)| = |\tilde{x}(t)| = \sqrt{x^2(t) + \hat{x}^2(t)} = \sqrt{x_I^2(t) + x_Q^2(t)} \\ \theta(t) &= \phi(t) - 2\pi f_0 t \\ \omega(t) &= \frac{d}{dt}\theta(t) \qquad\qquad f(t) = \frac{1}{2\pi}\omega(t) \end{aligned}$$

Properties

$$X(f) = \frac{X_I(f-f_0) + X_I(f+f_0)}{2} + \frac{X_Q(f-f_0) - X_Q(f+f_0)}{2j}$$

$$\begin{aligned} X_I(f) &= X^-(f-f_0) + X^+(f+f_0) \\ X_Q(f) &= \frac{X^-(f-f_0) - X^+(f+f_0)}{j} \end{aligned}$$

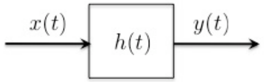
where $s^-(t) = s(t)u(-t)$ and $s^+(t) = s(t)u(t)$

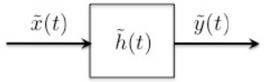
LTI and Low-Pass Representation

Consider a pass-band LTI system with impulse response $h(t)$

$$y(t) = x(t) \star h(t) = \int_R h(\tau)x(t-\tau)d\tau$$

where the input $x(t)$ and the output $y(t)$ are band-pass signals





The relation between the complex envelopes is

$$\begin{aligned}\tilde{y}(t) &= \mathring{y}(t)\exp(-j2\pi f_0 t) = \frac{\exp(-j2\pi f_0 t)}{2}\left(\mathring{x}(t) \star \mathring{h}(t)\right) \\ &= \frac{\exp(-j2\pi f_0 t)}{2}\left((\tilde{x}(t)\exp(+j2\pi f_0 t)) \star \left(\tilde{h}(t)\exp(+j2\pi f_0 t)\right)\right) \\ &= \frac{1}{2}\int_R \tilde{h}(\tau)\tilde{x}(t-\tau)d\tau = \frac{1}{2}\tilde{x}(t) \star \tilde{h}(t)\end{aligned}$$

WSS Band-Pass Random Signals (1/2)

Consider a WSS band-pass random signal $x(t)$, then

$$\begin{aligned}R_{\tilde{x}}(\tau) &= \mathbb{E}\{\tilde{x}(t)\tilde{x}^*(t-\tau)\} \\ &= \mathbb{E}\{\mathring{x}(t)\exp(-j2\pi f_0 t)\mathring{x}^*(t-\tau)\exp(+j2\pi f_0(t-\tau))\} \\ &= R_{\mathring{x}}(\tau)\exp(-j2\pi f_0\tau) \\ &= 2\left(R_x(\tau) + j\hat{R}_x(\tau)\right)\exp(-j2\pi f_0\tau) \\ &= 2\left(R_x(\tau)\cos(2\pi f_0\tau) + \hat{R}_x(\tau)\sin(2\pi f_0\tau)\right) \\ &\quad +j2\left(\hat{R}_x(\tau)\cos(2\pi f_0\tau) - R_x(\tau)\sin(2\pi f_0\tau)\right)\end{aligned}$$

i.e. the same relations refer now to the autocorrelation

$$R_{x_I}(\tau) = R_{x_Q}(\tau) = R_x(\tau)\cos(2\pi f_0\tau) + \hat{R}_x(\tau)\sin(2\pi f_0\tau)$$

$$R_{x_I x_Q}(\tau) = -R_{x_I x_Q}(-\tau) = -R_x(\tau)\sin(2\pi f_0\tau) + \hat{R}_x(\tau)\cos(2\pi f_0\tau)$$

WSS Band-Pass Random Signals (2/2)

Consider a WSS band-pass random signal $x(t)$, then

- the complex envelope $\tilde{x}(t)$ is WSS
- the I-Q components $x_I(t)$ and $x_Q(t)$ are jointly WSS and orthogonal
- the I-Q components $x_I(t)$ and $x_Q(t)$ are incoherent if the PSD of $x(t)$ is symmetric around f_0

and finally

$$R_x(\tau) = R_{x_I}(\tau)\cos(2\pi f_0\tau) + R_{x_Q}(\tau)\sin(2\pi f_0\tau)$$

An Interesting Example

Consider the following band-pass random signal

$$x(t) = A(t)\cos(2\pi f_0 t) + B(t)\sin(2\pi f_0 t)$$

where $A(t)$ and $B(t)$ are low-pass jointly WSS signals, then

- the signal is ciclostationary with average autocorrelation

$$R_x(\tau) = \frac{R_A(\tau) + R_B(\tau)}{2}\cos(2\pi f_0\tau) - \frac{R_{AB}(\tau) - R_{BA}(\tau)}{2}\sin(2\pi f_0\tau)$$

- the signal is WSS IFF $R_A(\tau) = R_B(\tau)$ and $R_{AB}(\tau) = -R_{AB}(-\tau)$