Analog Communications

— Lecture 05 —
Angle Modulation

Pierluigi SALVO ROSSI

Department of Industrial and Information Engineering
Second University of Naples
Via Roma 29, 81031 Aversa (CE), Italy

homepage: http://wpage.unina.it/salvoros
email: pierluigi.salvorossi@unina2.it

Outline

Angle Modulation

- $x(t)$ is the information low-pass signal or modulating signal
- $p(t) = A\cos(2\pi f_0 t + \vartheta)$ is the carrier signal
- the transmitted signal or modulated signal has the form

$$z(t) = A\cos(2\pi f_0 t + \psi_x(t))$$

  where $\psi_x(t)$ depends on $x(t)$ through linear transformations

- Phase Modulation (PM)

$$\psi_x(t) = s\ x(t)$$

- Frequency Modulation (FM)

$$\psi_x(t) = s\int_{t_0}^{t} x(\tau)d\tau$$

Instantaneous Frequency and Frequency Deviation

$$\begin{aligned} f_z(t) &= \frac{1}{2\pi}\frac{\partial}{\partial t}(2\pi f_0 t + \psi_x(t)) = f_0 + \frac{1}{2\pi}\frac{\partial}{\partial t}\psi_x(t) \\ \Delta f_z(t) &= \frac{1}{2\pi}\frac{\partial}{\partial t}\psi_x(t)\ , \quad \Delta f_{max} = \max_{t\in\mathbb{R}} |\Delta f_z(t)| \end{aligned}$$

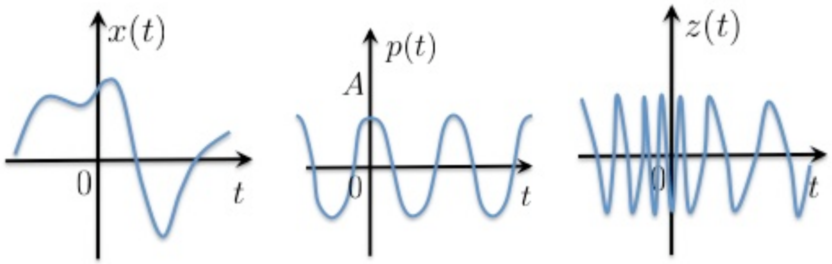
- PM

$$\begin{aligned} f_z(t) &= f_0 + \frac{s}{2\pi}\frac{\partial}{\partial t}x(t) \\ \Delta f_z(t) &= \frac{s}{2\pi}\frac{\partial}{\partial t}x(t)\ , \quad \Delta f_{max} = \frac{s}{2\pi}\left|\frac{\partial}{\partial t}x(t)\right|_{max} \end{aligned}$$

- FM

$$\begin{aligned} f_z(t) &= f_0 + \frac{s}{2\pi}x(t) \\ \Delta f_z(t) &= \frac{s}{2\pi}x(t)\ , \quad \Delta f_{max} = \frac{s}{2\pi}|x(t)|_{max} \end{aligned}$$

Signal Plots



Narrowband Approximation (1/2)

Using series expansion of exponential function

$$\begin{aligned} z(t) &= A\cos(2\pi f_0 t + \psi_x(t)) = A\Re\{\exp(j2\pi f_0 t)\exp(j\psi_x(t))\} \\ &= A\Re\left\{\exp(j2\pi f_0 t)\sum_{n=0}^{\infty}\frac{j^n\psi_x^n(t)}{n!}\right\} \end{aligned}$$

The bandwidth of $\psi_x^n(t)$ is n times the bandwidth of $\psi_x(t)$
The bandwidth of $z(t)$ is infinite

Assuming small phase variations, i.e. $|\psi_x(t)| \ll 1$

$$\begin{aligned} z(t) &\approx A\Re\{\exp(j2\pi f_0 t)(1 + j\psi_x(t))\} \\ &= A\cos(j2\pi f_0 t) - A\psi_x(t)\sin(j2\pi f_0 t) \end{aligned}$$

Looks like AM modulation:
the modulating signal is in quadrature w.r.t. the carrier signal

Correlation and PSD of Narrowband Angle Modulation

If $\psi_x(t)$ is a WSS signal with null expected value, then $z(t)$ is cyclostationary with period $1/2f_0$

$$\begin{aligned} R_z(t,\tau) &= \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{2}\cos(2\pi f_0(2t-\tau)) \\ &\quad + \frac{A^2}{2}R_{\psi_x}(\tau)\cos(2\pi f_0\tau) - \frac{A^2}{2}R_{\psi_x}(\tau)\cos(2\pi f_0(2t-\tau)) \\ R_z(\tau) &= \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{2}R_{\psi_x}(\tau)\cos(2\pi f_0\tau) \end{aligned}$$

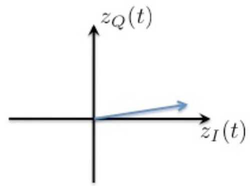
$$\begin{aligned} P_z(\tau) &= \frac{A^2}{4}(\delta(f-f_0) + \delta(f+f_0)) \\ &\quad + \frac{A^2}{4}(P_{\psi_x}(f-f_0) + P_{\psi_x}(f+f_0)) \\ P_z &= \frac{A^2}{2} + \frac{A^2}{2}P_{\psi_x} \\ B_z &= 2B_x \end{aligned}$$

Complex Envelope of Narrowband Angle Modulation

Analytic signal and complex envelope are

$$\begin{aligned} \mathring{z}(t) &= A(1 + j\psi_x(t))\exp(j2\pi f_0 t) \\ \tilde{z}(t) &= A(1 + j\psi_x(t)) \end{aligned}$$

- The complex envelope is complex-valued
- Real (resp. imaginary) component has arbitrary (resp. arbitrary small) value



Wideband Modulation: One Sinusoid Modulating

Focus on the FM case with modulating signal

$$x(t) = A_m \cos(2\pi f_m t + \theta_m)$$

then

$$\psi_x(t) = \frac{sA_m}{2\pi f_m} \sin(2\pi f_m t + \theta_m)$$

and the transmitted signal is

$$\begin{aligned} z(t) &= A \cos\left(2\pi f_0 t + \frac{sA_m}{2\pi f_m} \sin(2\pi f_m t + \theta_m)\right) \\ &= A \cos\left(2\pi f_0 t + \beta_m \sin(2\pi f_m t + \theta_m)\right) \end{aligned}$$

with

$$\Delta f_{max} = \frac{sA_m}{2\pi} \quad \text{and} \quad \beta_m = \frac{\Delta f_{max}}{f_m}$$

Wideband Modulation: Bessel Expansion (1/2)

$$\begin{aligned} z(t) &= A \cos\left(2\pi f_0 t + \beta_m \sin(2\pi f_m t + \theta_m)\right) \\ &= A \Re\left\{\exp(j 2\pi f_0 t) \exp(j\beta_m \sin(2\pi f_m t + \theta_m))\right\} \\ &= A \Re\left\{\exp(j 2\pi f_0 t) \sum_{n=-\infty}^{+\infty} J_n(\beta_m) \exp(jn\theta_m) \exp(j 2\pi n f_m t)\right\} \\ &= A \sum_{n=-\infty}^{+\infty} J_n(\beta_m) \cos\left(2\pi (f_0 + n f_m) t + n\theta_m\right) \end{aligned}$$

Infinite spectral components with frequency spacing equals to f_m

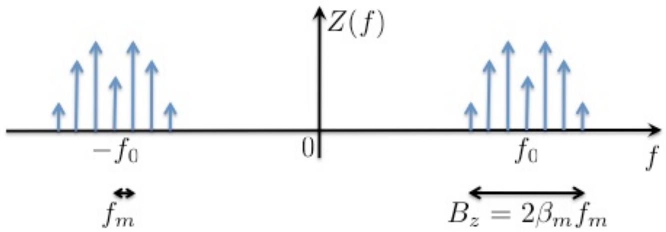
Wideband Modulation: Bessel Expansion (2/2)

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \exp(j(x \sin(\alpha) - n\alpha)) d\alpha = \frac{1}{\pi} \int_0^{+\pi} \cos(x \sin(\alpha) - n\alpha) d\alpha$$

- $J_0(0) = 1$ and $J_n(0) = 0$
- $J_{2n}(-x) = J_{2n}(x)$ and $J_{-2n}(x) = -J_{2n}(x)$
- $J_{2n-1}(-x) = -J_{2n-1}(x)$ and $J_{-2n+1}(x) = -J_{2n-1}(x)$
- $J_n(x) \approx 0$ if $n \gg x$
- $\sum_{n=-\infty}^{+\infty} J_n^2(x) = 1$

$$z(t) \approx A \sum_{n=-\beta_m}^{+\beta_m} J_n(\beta_m) \cos\left(2\pi (f_0 + n f_m) t + n\theta_m\right)$$

Wideband Modulation: Bandwidth



$$\begin{aligned} B_z &= 2\beta_m f_m \\ P_z &= \frac{A^2}{2} \end{aligned}$$

Wideband Modulation: Two Sinusoids Modulating (1/2)

Focus on the FM case with modulating signal with $f_1 > f_2$

$$x(t) = A_1 \cos(2\pi f_1 t + \theta_1) + A_2 \cos(2\pi f_2 t + \theta_2)$$

then

$$\psi_x(t) = \frac{sA_1}{2\pi f_1} \sin(2\pi f_1 t + \theta_1) + \frac{sA_2}{2\pi f_2} \sin(2\pi f_2 t + \theta_2)$$

and the transmitted signal is

$$z(t) = A \cos\left(2\pi f_0 t + \beta_1 \sin(2\pi f_1 t + \theta_1) + \beta_2 \sin(2\pi f_2 t + \theta_2)\right)$$

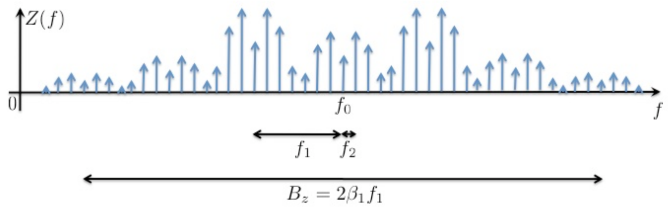
with

$$\beta_1 = \frac{sA_1}{2\pi f_1} \quad \text{and} \quad \beta_2 = \frac{sA_2}{2\pi f_2}$$

Wideband Modulation: Two Sinusoids Modulating (2/2)

Using twice the Bessel expansion

$$z(t) \approx A \sum_{n_1=-\beta_1}^{+\beta_1} \sum_{n_2=-\beta_2}^{+\beta_2} J_{n_1}(\beta_1) J_{n_2}(\beta_2) \times \cos\left(2\pi(f_0 + n_1 f_{m_1} + n_2 f_{m_2})t + n_1\theta_1 + n_2\theta_2\right)$$



$$B_z = 2\beta_1 f_1 \quad \text{and} \quad P_z = \frac{A^2}{2}$$

Wideband Modulation: N Sinusoids Modulating

Focus on the FM case with modulating signal

$$x(t) = \sum_{\ell=1}^{N} A_\ell \cos(2\pi f_\ell t + \theta_\ell) \quad \text{with} \quad \beta_\ell = \frac{sA_\ell}{2\pi f_\ell}$$

then

$$z(t) \approx A \sum_{n_1=-\beta_1}^{+\beta_1} \cdots \sum_{n_N=-\beta_N}^{+\beta_N} J_{n_1}(\beta_1) \cdots J_{n_N}(\beta_N) \times \cos\left(2\pi\left(f_0 + \sum_{\ell=1}^{N} n_\ell f_\ell\right)t + \sum_{\ell=1}^{N} n_\ell \theta_\ell\right)$$

$$m = \arg \max_{\ell=1,\dots,N} f_\ell$$

$$B_z = 2\beta_m f_m \quad \text{and} \quad P_z = \frac{A^2}{2}$$

Carson's Bandwidth Rule

Assume a low-pass modulating signal $x(t)$ with

- B_x is the bandwidth of $x(t)$
- $A_x = \max_t |x(t)|$ is the maximum amplitude
- $\beta = \frac{sA_x}{2\pi B_x} = \frac{\Delta f_{max}}{B_x}$

then extending the previous considerations we get

$$B_z = 2\beta B_x = 2\Delta f_{max}$$

Merging results for narrowband modulation ($\beta \ll 1$) and wideband modulation ($\beta \gg 1$), we get

$$B_z = 2(1 + \beta)B_x$$

Wideband Modulation and Woodward's Theorem (1/3)

Consider a II-order stationary random process $\psi_x(t)$ as modulating signal

$$
\begin{aligned}
R_z(t,\tau) &= A^2\mathbb{E}\left\{\cos(2\pi f_0 t + \psi_x(t))\cos(2\pi f_0(t-\tau)+\psi_x(t-\tau))\right\} \\
&= \frac{A^2}{2}\mathbb{E}\left\{\cos(2\pi f_0(2t-\tau)+\psi_x(t)+\psi_x(t-\tau))\right\} \\
&\quad +\frac{A^2}{2}\mathbb{E}\left\{\cos(2\pi f_0(\tau)+\psi_x(t)-\psi_x(t-\tau))\right\} \\
&= \frac{A^2}{2}\Re\left[e^{j2\pi f_0 2t}e^{-j2\pi f_0\tau}\mathbb{E}\{e^{j\psi_x(t)}e^{j\psi_x(t-\tau)}\}\right] \\
&\quad +\frac{A^2}{2}\Re\left[e^{j2\pi f_0\tau}\mathbb{E}\{e^{j\psi_x(t)}e^{-j\psi_x(t-\tau)}\}\right] \\
&= \frac{A^2}{2}\Re\left[e^{j2\pi f_0 2t}e^{-j2\pi f_0\tau}\mathbb{E}\{y(t)y(t-\tau)\}\right] \\
&\quad +\frac{A^2}{2}\Re\left[e^{j2\pi f_0\tau}\mathbb{E}\{y(t)y^*(t-\tau)\}\right]
\end{aligned}
$$

where $y(t)=\exp(j\psi_x(t))$

Wideband Modulation and Woodward's Theorem (2/3)

- The II-order stationarity assumption makes both $\mathbb{E}\{y(t)y(t-\tau)\}$ and $\mathbb{E}\{y(t)y^*(t-\tau)\}=R_y(\tau)$ independent on t
- $z(t)$ is cyclostationary with period $1/2f_0$

$$
\begin{aligned}
R_z(\tau) &= \frac{A^2}{2}\Re\left[e^{j2\pi f_0\tau}R_y(\tau)\right] \\
&= \frac{A^2}{4}\left(e^{j2\pi f_0\tau}R_y(\tau)+e^{-j2\pi f_0\tau}R_y(-\tau)\right)
\end{aligned}
$$

and

$$P_z(f)=\frac{A^2}{4}\left(P_y(f-f_0)+P_y(-f-f_0)\right)$$

Wideband Modulation and Woodward's Theorem (3/3)

Assume $\psi_x(t)$ slowly changing, then $\psi_x(t)-\psi_x(t-\tau)\approx\tau\dot{\psi}_x(t)$ and

$$
\begin{aligned}
R_y(\tau) &= \mathbb{E}\left\{e^{j\tau\dot{\psi}_x(t)}\right\}=\int_{\mathbb{R}}e^{j\tau\xi}f_{\dot{\psi}_x}(\xi)d\xi \\
&= 2\pi\mathcal{F}^{-1}\left\{f_{\dot{\psi}_x}(2\pi f)\right\}(\tau)
\end{aligned}
$$

then $P_y(f)=2\pi f_{\dot{\psi}_x}(2\pi f)$, giving

$$P_z(f)=\frac{\pi A^2}{2}\left(f_{\dot{\psi}_x}(2\pi(f-f_0))+f_{\dot{\psi}_x}(-2\pi(f+f_0))\right), \text{and} \quad P_z=\frac{A^2}{2}$$

In the case of FM we have the Woodward's Theorem

$$P_z(f)=\frac{\pi A^2}{2s}\left(f_x\left(\frac{2\pi}{s}(f-f_0)\right)+f_x\left(-\frac{2\pi}{s}(f+f_0)\right)\right)$$

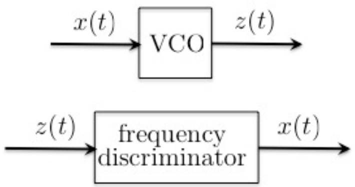
Bandwidth

- The shape of the PSD of the modulated signal $z(t)$ follows the shape of the pdf of the modulating signal $x(t)$
- The bandwidth B_z is approximatively given by the support of the pdf $f_x(\cdot)$
- The support is proportional to the standard deviation σ_x

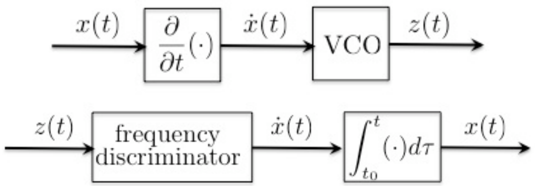
The bandwidth is proportional to the amplitude of the modulating signal

Modulation and Demodulation

FM modulation and demodulaton



PM modulation and demodulation



Frequency Discriminator

$$
\begin{aligned}
\psi_x(t) &= s\int_{t_0}^{t} x(\tau)d\tau \\
z(t) &= A\cos\left(2\pi f_0 t + \psi_x(t)\right) \\
\dot{z}(t) &= -A(2\pi f_0 + \dot{\psi}_x(t))\sin\left(2\pi f_0 t + \psi_x(t)\right)
\end{aligned}
$$

Assuming $f_0 \geq \frac{s}{2\pi}|x(t)|_{max}$ we get

$$
\begin{aligned}
v(t) &= A(2\pi f_0 + sx(t)) \\
y(t) &= Asx(t)
\end{aligned}
$$