Digital Communications

— Lecture 02 —
Signal-Space Representation

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Outline

Energy Signals

$\mathcal{L}^2(\mathbb{D})$ is the set of the energy signals over $\mathbb{D} \subseteq \mathbb{R}$, i.e.

$$s(t) \in \mathcal{L}^2(\mathbb{D}) \quad \leftrightarrow \quad \mathcal{E}_s \triangleq \int_{\mathbb{D}} |s(t)|^2 \, dt < +\infty$$

$\mathcal{L}^2(\mathbb{D})$ is a Hilbert space with (scalar) inner product

$$< s_1, s_2 >\triangleq \int_{\mathbb{D}} s_1(t) s_2^*(t) dt \quad \forall\, s_1(t), s_2(t) \in \mathcal{L}^2(\mathbb{D})$$

$\{\psi_n(t)\}_{n=1}^N$ is said an orthonormal set of signals in $\mathcal{L}^2(\mathbb{D})$ if

$$< \psi_n, \psi_m >= \int_{\mathbb{D}} \psi_n(t) \psi_m^*(t) dt = \delta_{n,m}$$

MMSE Approximation

Given an orthonormal set of signals $\{\psi_n(t)\}_{n=1}^N$ in $\mathcal{L}^2(\mathbb{D})$, which one is the best linear combination to represent a generic signal $s(t) \in \mathcal{L}^2(\mathbb{D})$?

Define the generic approximation

$$\hat{s}(t) \triangleq \sum_{n=1}^{N} c_n \psi_n(t)$$

the usual criterion to find the best vector of coefficients $\boldsymbol{c} = (c_1, \ldots, c_n)^T$ is the Minimum Mean Square Error (MMSE) criterion

Define the error signal and the Mean Square Error (MSE), i.e. its energy

$$\begin{aligned} e(t) &\triangleq s(t) - \hat{s}(t) \\ \epsilon_{\text{rms}}^2 &\triangleq \int_{\mathbb{D}} |e(t)|^2 \, dt \end{aligned}$$

MSE

The MSE can be expressed as

$$\begin{aligned}
\epsilon^2_{\text{rms}} &= \int_{\mathbb{D}} |s(t)|^2\, dt + \int_{\mathbb{D}} |\hat{s}(t)|^2\, dt - 2\Re\left\{\int_{\mathbb{D}} s(t)\hat{s}^*(t)dt\right\} \\
&= \mathcal{E}_s + \sum_{n=1}^{N} |c_n|^2 - 2\Re\left\{\sum_{n=1}^{N} c_n^* \int_{\mathbb{D}} s(t)\psi_n^*(t)dt\right\} \\
&= \mathcal{E}_s - \sum_{n=1}^{N}\left|\int_{\mathbb{D}} s(t)\psi_n^*(t)dt\right|^2 + \underbrace{\sum_{n=1}^{N}\left|c_n - \int_{\mathbb{D}} s(t)\psi_n^*(t)dt\right|^2}_{\geq 0}
\end{aligned}$$

The last term is the only depending on $\boldsymbol{c}$ thus the MMSE is achieved when such a term is null

Constellation Point

Denote $\boldsymbol{s} = (s_1, \ldots, s_N)^T$ the coefficient vector achieving the MMSE

$$\boldsymbol{s} = \arg\min_{\boldsymbol{c}\in\mathbb{C}^N} \epsilon^2_{\text{rms}}$$

Such vector is also called the constellation point in the signal space and its component are computed as

$$s_n = < s, \psi_n > = \int_{\mathbb{D}} s(t)\psi_n^*(t)dt$$

The corresponding MMSE is

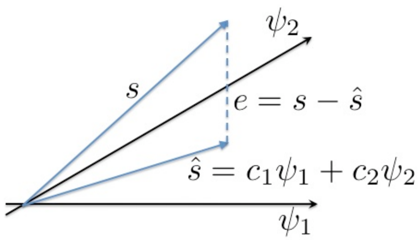
$$\begin{aligned}
\epsilon^2_{\text{rms},opt} &= \min_{\boldsymbol{c}\in\mathbb{C}^N} \epsilon^2_{\text{rms}} \\
&= \mathcal{E}_s - \sum_{n=1}^{N} |s_n|^2
\end{aligned}$$

Orthogonality Principle

It is a graphical interpretation of the MMSE result

$$\begin{aligned}
< e, \psi_n > &= < s - \hat{s}, \psi_n > \\
&= < s, \psi_n > - < \hat{s}, \psi_n > \\
&= s_n - s_n \\
&= 0
\end{aligned}$$

Error ($e(t)$) and data ($\{\psi_n(t)\}_{n=1}^{N}$) are orthogonal



Gram-Schmidt Process

It takes a set of signals $\{s_m(t)\}_{m=1}^{M}$ spanning a subset of $\mathcal{L}^2(\mathbb{D})$ and provides an orthogonal set $\{\psi_n(t)\}_{n=1}^{N}$, with $N \leq M$, spanning the same (N-dimensional) subset

Iterate the following:

$$\begin{aligned}
\tilde{\psi}_n(t) &= s_n(t) - \sum_{\ell=1}^{n-1} < s_n, \psi_\ell > \psi_\ell(t) \\
\psi_n(t) &= \frac{\tilde{\psi}_n(t)}{\sqrt{< \tilde{\psi}_n, \tilde{\psi}_n >}}
\end{aligned}$$

Signal Constellation

A set of signals $\{s_m(t)\}_{m=1}^{M}$ can be described through an orthogonal set $\{\psi_n(t)\}_{n=1}^{N}$, with $N \leq M$, via a set of M (N-dimensional) vectors $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$ denoted signal constellation

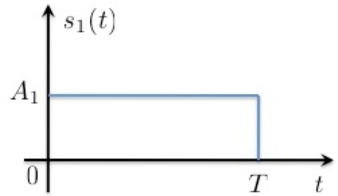
The mth vector (or constellation point) associated to $s_m(t)$ is

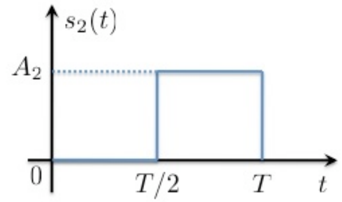
$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,n} \\ \vdots \\ s_{m,N} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_n > \\ \vdots \\ < s_m, \psi_N > \end{pmatrix}$$

Example 1 (1/6)

Consider a binary modulation ($M = 2$) with signals

$$s_1(t) = A_1 \text{ rect}\left(\frac{t - T/2}{T}\right) \qquad s_2(t) = A_2 \text{ rect}\left(\frac{t - 3T/4}{T/2}\right)$$



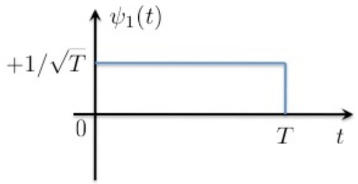


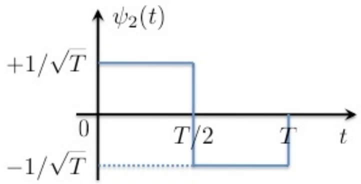
The signals have the following energies: $\mathcal{E}_1 = A_1^2 T$ and $\mathcal{E}_2 = A_2^2 T/2$

Example 1 (2/6)

A possible orthonormal set of signals is

$$\psi_1(t) = \frac{1}{\sqrt{T}} \text{rect}\left(\frac{t - T/2}{T}\right)$$
$$\psi_2(t) = \frac{1}{\sqrt{T}} \left(\text{rect}\left(\frac{t - T/4}{T/2}\right) - \text{rect}\left(\frac{t - 3T/4}{T/2}\right)\right)$$

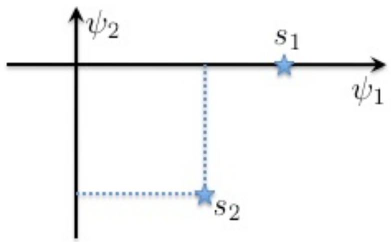




Example 1 (3/6)

The constellation is the following

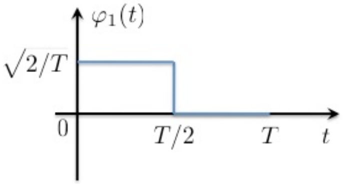
$$\boldsymbol{s}_1 = \begin{pmatrix} +A_1\sqrt{T} \\ 0 \end{pmatrix} \qquad \boldsymbol{s}_2 = \begin{pmatrix} +A_2\sqrt{T}/2 \\ -A_2\sqrt{T}/2 \end{pmatrix}$$

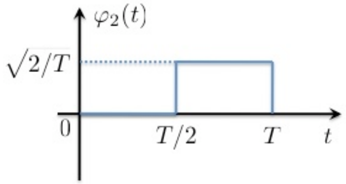


Example 1 (4/6)

Another possible orthonormal set of signals is

$$\varphi_1(t) = \sqrt{\frac{2}{T}}\text{rect}\left(\frac{t-T/4}{T/2}\right)$$
$$\varphi_2(t) = \sqrt{\frac{2}{T}}\text{rect}\left(\frac{t-3T/4}{T/2}\right)$$

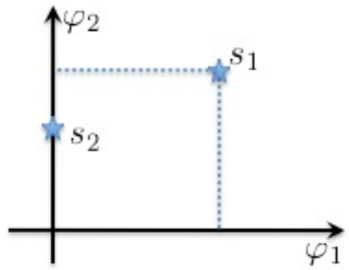




Example 1 (5/6)

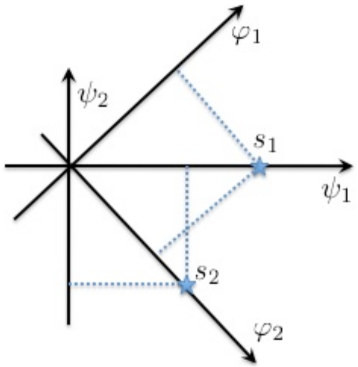
The constellation is the following

$$\boldsymbol{s}_1 = \begin{pmatrix} +A_1\sqrt{T/2} \\ +A_1\sqrt{T/2} \end{pmatrix} \qquad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ +A_2\sqrt{T/2} \end{pmatrix}$$



Example 1 (6/6)

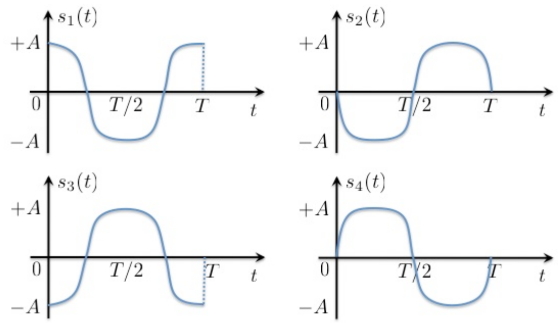
Selecting a different orthonormal set of signals for representation corresponds to a rotation of the reference axis of the signal space



Example 2 - QPSK (1/3)

Consider a quaternary modulation ($M = 4$) with signals

$$s_m(t) = A\cos\left(\frac{2\pi}{T}t + (m-1)\frac{\pi}{2}\right)\text{rect}\left(\frac{t-T/2}{T}\right) \qquad m = 1,2,3,4$$

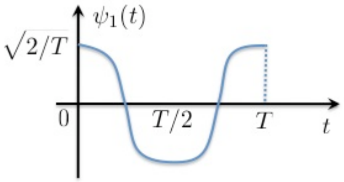


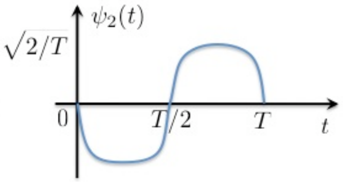
The signals have all the same energy $\mathcal{E} = A^2T/2$

Example 2 - QPSK (2/3)

A possible orthonormal set of signals is

$$\psi_1(t) = +\sqrt{\frac{2}{T}}\cos\left(\frac{2\pi}{T}t\right)\text{rect}\left(\frac{t-T/2}{T}\right)$$
$$\psi_2(t) = -\sqrt{\frac{2}{T}}\sin\left(\frac{2\pi}{T}t\right)\text{rect}\left(\frac{t-T/2}{T}\right)$$

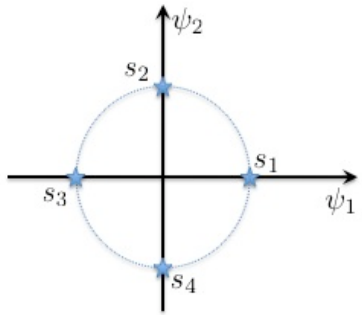




Example 2 - QPSK (3/3)

The constellation is the following

$$\boldsymbol{s}_1 = \begin{pmatrix} +A\sqrt{T/2} \\ 0 \end{pmatrix} \qquad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ +A\sqrt{T/2} \end{pmatrix}$$
$$\boldsymbol{s}_3 = \begin{pmatrix} -A\sqrt{T/2} \\ 0 \end{pmatrix} \qquad \boldsymbol{s}_4 = \begin{pmatrix} 0 \\ -A\sqrt{T/2} \end{pmatrix}$$



Representation of Stochastic Processes (1/2)

Consider a stochastic process $x(t) \in \mathcal{L}^2(\mathbb{D})$, i.e. such that each realization is an energy signal over $\mathbb{D}$, and an orthonormal set of signals $\{\psi_n(t)\}_{n=1}^{\infty}$

$$x(t) = \sum_{n=1}^{\infty} x_n \psi_n(t)$$
$$x_n = < x, \psi_n >$$

The dimension is ∞ in order to represent each possible realization

The stochastic process is represented by a random vector with infinite components

$$x(t) \quad \leftrightarrow \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$$

Representation of Stochastic Processes (2/2)

Each component $x_n = \int_{\mathbb{D}} x(t)\psi_n^*(t)dt$ is a random variable

$$\mathbb{E}\{x_n\} = \mathbb{E}\left\{\int_{\mathbb{D}} x(t)\psi_n^*(t)dt\right\} = \int_{\mathbb{D}} \mu_x(t)\psi_n^*(t)dt$$

$$\begin{aligned} & \text{Cov}\{x_n, x_m\} = \mathbb{E}\{(x_n - \mathbb{E}\{x_n\})(x_m - \mathbb{E}\{x_m\})^*\} \\ = \; & \mathbb{E}\left\{\iint_{\mathbb{D}^2} (x(t) - \mu_x(t))(x(s) - \mu_x(s))^* \psi_n^*(t)\psi_m(s)dtds\right\} \\ = \; & \iint_{\mathbb{D}^2} K_x(t,s)\psi_n^*(t)\psi_m(s)dtds \end{aligned}$$

For a WSS process $\mu_x(t) = \mu_x$ and $K_x(t,s) = R_x(t-s) - |\mu_x|^2$

WGN Charachterization

Consider a White Gaussian Noise (WGN) $w(t) \in \mathcal{L}^2(\mathbb{D})$ with zero-mean independent real and imaginary parts, each with $\eta_0/2$ flat PSD

Each component $w_n = \int_{\mathbb{D}} w(t)\psi_n^*(t)dt$ is a complex Gaussian r.v.

$$
\begin{aligned}
\mathbb{E}\{w_n\} &= 0 \\
\mathbb{C}\text{ov}\{w_n, w_m\} &= \iint_{\mathbb{D}^2} \eta_0 \delta(t-s)\psi_n^*(t)\psi_m(s)dtds \\
&= \eta_0 \int_{\mathbb{D}} \psi_n^*(t)\psi_m(t)dt = \eta_0 \delta_{n,m}
\end{aligned}
$$

$\boldsymbol{w}$ is a complex Gaussian random vector with uncorrelated thus independent components

$$
w_n \sim \mathcal{N}_{\mathbb{C}}(0, \eta_0) \quad \leftrightarrow \quad \Re\{w_n\}, \Im\{w_n\} \sim \mathcal{N}(0, \eta_0/2)
$$