Digital Communications

— Lecture 03 —

On-Off Keying

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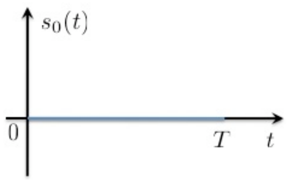
Outline

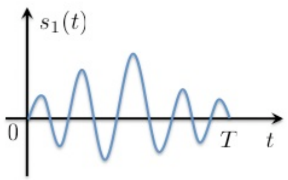
System Model

Binary memoryless modulation: $M = 2$, $\mathcal{A} = \{0, 1\}$, $\Pi = \{p, 1-p\}$

Consider real-valued signals and noise

$$s_0(t) = 0 \qquad s_1(t) = p(t) \qquad t \in [0, T)$$





AWGN channel: $\mu_w(t) = 0$, $P_w(f) = \eta_0/2$

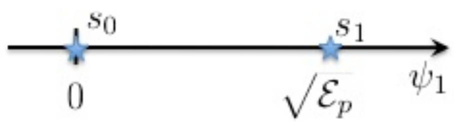
Signal Constellation

The signal space has dimension $N = 1$

$$\psi_1(t) = \frac{1}{\sqrt{\mathcal{E}_p}} p(t)$$
$$\mathcal{E}_p = \int_0^T p^2(t)dt$$

The signal constellation is

$$\boldsymbol{s}_0 = < s_0, \psi_1 > = 0$$
$$\boldsymbol{s}_1 = < s_1, \psi_1 > = \sqrt{\mathcal{E}_p}$$



Received Signal

Two possible hypotheses in $t \in [0, T)$

$$H_0 \quad : \quad r(t) = w(t)$$
$$H_1 \quad : \quad r(t) = p(t) + w(t)$$

To represent the noise, thus the received signal, we need an infinite orthonormal set $\{\psi_n(t)\}_{n=1}^{\infty}$. Assume that $\psi_1(t) = p(t)/\sqrt{\mathcal{E}_p}$

$$\boldsymbol{r}|H_0 = \begin{pmatrix} < w, \psi_1 > \\ < w, \psi_2 > \\ \vdots \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

$$\boldsymbol{r}|H_1 = \begin{pmatrix} < s + w, \psi_1 > \\ < s + w, \psi_2 > \\ \vdots \end{pmatrix} = \begin{pmatrix} \sqrt{\mathcal{E}_p} \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

Sufficient Statistic (1/2)

Two considerations are crucial:

- Only the first component of the received vector is dependent on the two hypotheses
- The components of the received vector are statistically independent

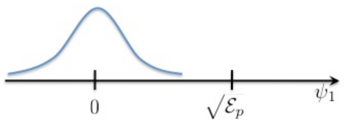
The first component r_1, or simply r, represents a sufficient statistic for the considered detection problem, thus is the only to be computed

$$r|H_0 = w_1 \qquad \sim \mathcal{N}\left(0, \frac{\eta_0}{2}\right)$$
$$r|H_1 = \sqrt{\mathcal{E}_p} + w_1 \qquad \sim \mathcal{N}\left(\sqrt{\mathcal{E}_p}, \frac{\eta_0}{2}\right)$$

Sufficient Statistic (2/2)

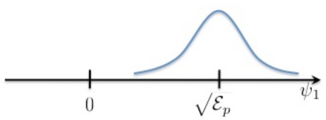
When transmitting the symbol 0 we observe a random variable with pdf

$$f_{r|H_0}(r) = \frac{1}{\sqrt{\pi\eta_0}} \exp\left(-\frac{r^2}{\eta_0}\right)$$



When transmitting the symbol 1 we observe a random variable with pdf

$$f_{r|H_1}(r) = \frac{1}{\sqrt{\pi\eta_0}} \exp\left(-\frac{(r - \sqrt{\mathcal{E}_p})^2}{\eta_0}\right)$$



Optimum Decision (1/2)

Decision is done via $\{\Omega_0, \Omega_1\}$ representing a partition of $\mathbb{R}$, i.e. $\Omega_0 \bigcup \Omega_1 = \mathbb{R}$ and $\Omega_0 \bigcap \Omega_1 = \emptyset$, with

$$r \in \Omega_0 \rightarrow \text{rx } 0 \qquad , \qquad r \in \Omega_1 \rightarrow \text{rx } 1$$

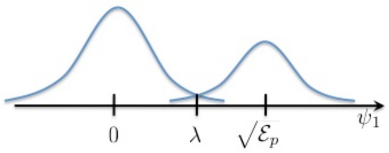
Each partition leads to a different error probability

$$\begin{aligned} P_e &\triangleq \Pr\left\{(\text{tx } 0, \text{ rx } 1) \bigcup (\text{tx } 1, \text{ rx } 0)\right\} \\ &= \Pr\{(\text{rx } 1|\text{tx } 0)\} p + \Pr\{(\text{rx } 0|\text{tx } 1)\}(1-p) \\ &= \Pr\{r|H_0 \in \Omega_1\} p + \Pr\{r|H_1 \in \Omega_0\}(1-p) \\ &= \int_{\Omega_1} p f_{r|H_0}(r) dr_1 + \int_{\Omega_0} (1-p) f_{r|H_1}(r) dr_1 \\ &= \int_{\mathbb{R}-\Omega_0} p f_{r|H_0}(r) dr_1 + \int_{\Omega_0} (1-p) f_{r|H_1}(r) dr_1 \\ &= p - \int_{\Omega_0} \left(p f_{r|H_0}(r) - (1-p) f_{r|H_1}(r)\right) dr_1 \end{aligned}$$

Optimum Decision (2/2)

The optimum decision leads to the minimum $\Pr(e)$, thus

$$\begin{aligned}\Omega_0 &= \{r \in \mathbb{R} : p f_{r|H_0}(r) > (1-p) f_{r|H_1}(r)\} \\ \Omega_1 &= \{r \in \mathbb{R} : p f_{r|H_0}(r) < (1-p) f_{r|H_1}(r)\}\end{aligned}$$



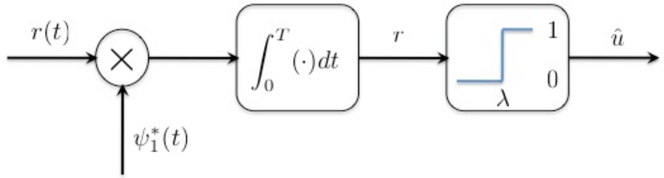
$$\begin{aligned}\Omega_0 &= \{r \in \mathbb{R} : r < \lambda\} \\ \Omega_1 &= \{r \in \mathbb{R} : r > \lambda\}\end{aligned}$$

with

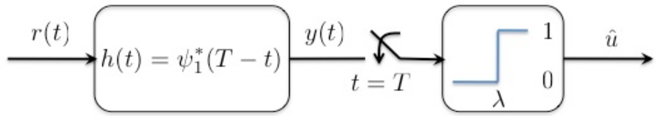
$$\lambda : \qquad p f_{r|H_0}(\lambda) = (1-p) f_{r|H_1}(\lambda)$$

Receiver Architecture (1/2)

First option

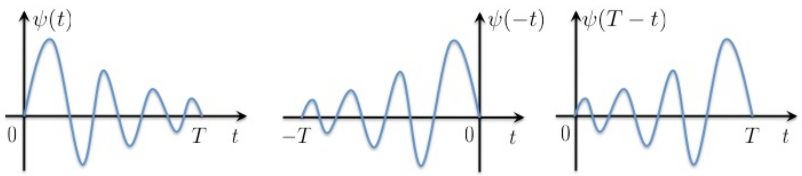


Second option



Receiver Architecture (2/2)

$$\begin{aligned}y(t) &= r(t) \star h(t) = \int_{-\infty}^{+\infty} r(\tau) h(t-\tau) d\tau \\ &= \int_0^t r(\tau) \psi_1(T - t + \tau) d\tau\end{aligned}$$



$$y(t)|_{t=T} = \int_0^T r(\tau) \psi_1(\tau) d\tau = r$$

Optimum Threshold

$$\begin{aligned}\lambda : \qquad p f_{r|H_0}(\lambda) &= (1-p) f_{r|H_1}(\lambda) \\ \frac{p}{\sqrt{\pi \eta_0}} \exp\left(-\frac{\lambda^2}{\eta_0}\right) &= \frac{1-p}{\sqrt{\pi \eta_0}} \exp\left(-\frac{\left(\lambda - \sqrt{\mathcal{E}_p}\right)^2}{\eta_0}\right)\end{aligned}$$

The optimum threshold depends on the energy, on the noise PSD and on the *a-priori* probabilities:

$$\lambda = \frac{\sqrt{\mathcal{E}_p}}{2} + \frac{\eta_0}{2\sqrt{\mathcal{E}_p}} \log\left(\frac{p}{1-p}\right)$$

In case of transmission with equiprobable symbols, i.e. $p = 1/2$, the optimum decision is a minimum distance decision:

$$\lambda = \frac{\sqrt{\mathcal{E}_p}}{2}$$

Error Probability (1/2)

$$
\begin{aligned}
P_e &= 1 - P_c = 1 - p\,\Pr(c|H_0) - (1-p)\,\Pr(c|H_1) \\
&= 1 - p\,\Pr(r|H_0 < \lambda) - (1-p)\,\Pr(r|H_1 > \lambda) \\
&= 1 - p\int_{-\infty}^{\lambda} f_{r|H_0}(r)dr - (1-p)\int_{\lambda}^{+\infty} f_{r|H_1}(r)dr \\
&= 1 - p\,\Pr\left(\mathcal{N}\left(0,\frac{\eta_0}{2}\right) < \lambda\right) - (1-p)\,\Pr\left(\mathcal{N}\left(\sqrt{\mathcal{E}_p},\frac{\eta_0}{2}\right) > \lambda\right) \\
&= 1 - p\left(1 - Q\left(\frac{\lambda}{\sqrt{\eta_0/2}}\right)\right) - (1-p)Q\left(\frac{\lambda-\sqrt{\mathcal{E}_p}}{\sqrt{\eta_0/2}}\right)
\end{aligned}
$$

In case of equiprobable symbols

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)$$

Error Probability (2/2)