Digital Communications

— Lecture 04 —
M-ary Digital Memoryless Modulation
over AWGN Channel

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Outline

System Model

- $\mathcal{A} = \{a_1, \dots, a_M\}$
- $\Pi = \{p_1, \dots, p_M\}$
- $\{s_1(t), \dots, s_M(t)\},\ t \in [0, T)$
- $\mathcal{E}_m$ is the energy of the mth signal

Consider real-valued signals and noise

$$u = a_m \quad \rightarrow \quad s_m(t) \in \mathbb{R} \quad \text{is trasmitted}$$

AWGN channel: $\mu_w(t) = 0$, $P_w(f) = \eta_0/2$, $w(t) \in \mathbb{R}$

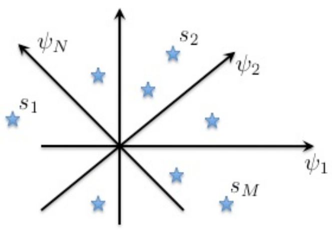
The signal space has dimension $N \leq M$ and $\{\psi_1(t), \dots, \psi_N(t)\}, t \in [0, T)$ is an orthonormal set for signal representation

Signal Constellation

The mth signal $s_m(t)$ is represented by

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,N} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_N > \end{pmatrix}$$

The signal constellation is $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_M\}$



Received Signal

M possible hypotheses in $t \in [0, T)$

$$H_m \quad : \quad r(t) = s_m(t) + w(t) \qquad m = 1, \dots, M$$

To represent the noise and the received signal, we need an infinite orthonormal set $\{\psi_n(t)\}_{n=1}^{\infty}$.

Assume that the first N signals in the orthonormal set are those used for signal space representation

$$\boldsymbol{r}|H_m = \begin{pmatrix} < s_m + w, \psi_1 > \\ \vdots \\ < s_m + w, \psi_N > \\ < s_m + w, \psi_{N+1} > \\ \vdots \end{pmatrix} = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,N} \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_N \\ w_{N+1} \\ \vdots \end{pmatrix}$$

Sufficient Statistic (1/2)

Two considerations are crucial:

- Only the first N components of the received vector are dependent on the M hypotheses
- The components of the received vector are statistically independent

The first N components $r_1, \dots, r_N$, or simply the N-dimensional vector $\boldsymbol{r} = (r_1, \dots, r_N)^T$, represent a sufficient statistic for the considered detection problem, thus are the only to be computed

$$\boldsymbol{r}|H_m = \boldsymbol{s}_m + \boldsymbol{w} \qquad \sim \mathcal{N}\left(\boldsymbol{s}_m, \frac{\eta_0}{2}\boldsymbol{I}_N\right)$$

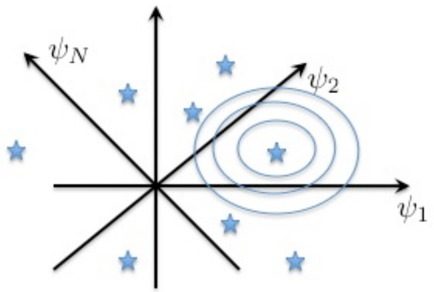
where the noise vector is $\boldsymbol{w} = (w_1, \dots, w_N)^T$

Sufficient Statistic (2/2)

When transmitting the symbol a_m we observe a random vector with N-dimensional pdf

$$f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) = \left(\frac{1}{\pi\eta_0}\right)^{N/2} \exp\left(-\frac{\|\boldsymbol{r} - \boldsymbol{s}_m\|^2}{\eta_0}\right)$$

$$= \left(\frac{1}{\pi\eta_0}\right)^{N/2} \exp\left(-\frac{1}{\eta_0}\sum_{n=1}^{N}(r_n - s_{m,n})^2\right)$$



Optimum Decision (1/2)

Decision is done via a partition of $\mathbb{R}^N$, i.e. $\Omega_1, \dots, \Omega_M$:

$$\begin{cases} \Omega_m \bigcap \Omega_\ell = \emptyset & \forall \ell \neq m \\ \bigcup_{m=1}^{M} \Omega_m = \mathbb{R}^N \end{cases}$$

with

$$\boldsymbol{r} \in \Omega_m \qquad \rightarrow \qquad \text{rx } a_m$$

Each partition leads to a different error probability

$$P_e \triangleq 1 - P_c = 1 - \sum_{m=1}^{M} p_m \Pr(c|H_m)$$

$$= 1 - \sum_{m=1}^{M} p_m \Pr(\boldsymbol{r}|H_m \in \Omega_m)$$

$$= 1 - \sum_{m=1}^{M} p_m \int_{\Omega_m} f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) d\boldsymbol{r}$$

Optimum Decision (2/2)

The optimum decision is the one corresponding to the minimum $\Pr(e)$, thus associated to the following partition

$$\Omega_m = \{\boldsymbol{r} \in \mathbb{R}^N : p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) > p_\ell f_{\boldsymbol{r}|H_\ell}(\boldsymbol{r}), \forall \ell \neq m\}$$

i.e. the decision rule is the following

$$\begin{aligned}\hat{u} &= \arg\max_m \{p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r})\} \\ &= \arg\max_m \left\{\frac{p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r})}{f_{\boldsymbol{r}}(\boldsymbol{r})}\right\} \\ &= \arg\max_m \{\Pr(H_m|\boldsymbol{r})\}\end{aligned}$$

The last equality explains the name of Maximum *A-posteriori* Probability (MAP) decision

MAP vs ML

The MAP decision rule provides the minimum error probability

$$\hat{u} = \arg\max_m \{p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r})\}$$

If the a-priori probabilities are all equal ($p_m = 1/M$), the rules becomes

$$\hat{u} = \arg\max_m \{f_{\boldsymbol{r}|H_m}(\boldsymbol{r})\}$$

usually named Maximum Likelihood (ML) decision rule

ML is also applied when a-priori probabilities are unknown, however in such cases minimum error probability is not guaranteed

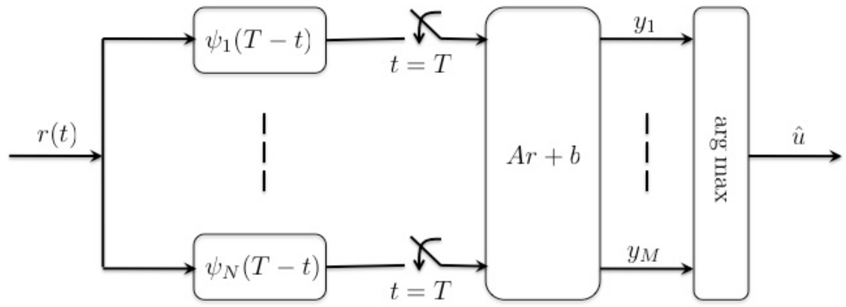
MAP decision (1/2)

$$\begin{aligned}\hat{u} &= \arg\max_m \{p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r})\} \\ &= \arg\max_m \left\{\frac{p_m}{(\pi\eta_0)^{N/2}} \exp\left(-\frac{1}{\eta_0}\sum_{n=1}^{N}(r_n - s_{m,n})^2\right)\right\} \\ &= \arg\max_m \left\{\eta_0 \log(p_m) - \sum_{n=1}^{N}(r_n - s_{m,n})^2\right\} \\ &= \arg\max_m \left\{\frac{\eta_0}{2}\log(p_m) - \frac{\|\boldsymbol{s}_m\|^2}{2} + \boldsymbol{s}_m^T \boldsymbol{r}\right\} \\ &= \arg\max_m \{y_m\}\end{aligned}$$

where defining $\boldsymbol{A} = (\boldsymbol{s}_1^T, \ldots, \boldsymbol{s}_M^T)^T$ and $b_m = \frac{\eta_0}{2}\log(p_m) - \frac{\|\boldsymbol{s}_m\|^2}{2}$ we denote

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{r} + \boldsymbol{b}$$

MAP Receiver Architecture (1/2)



MAP decision (2/2)

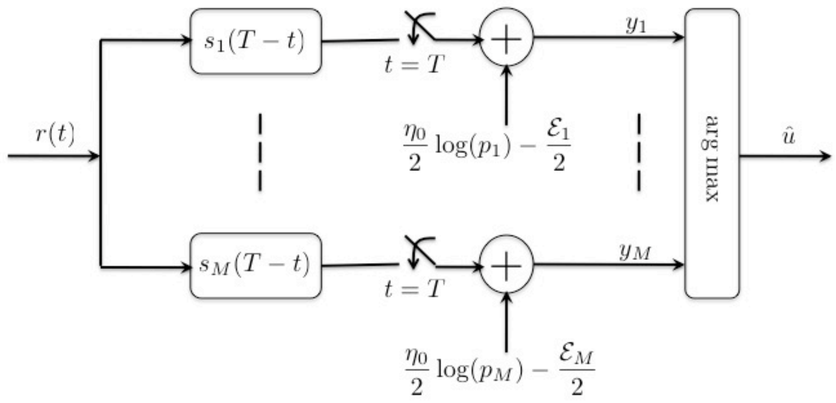
It is worth noticing that

$$
\begin{aligned}
\mathcal{E}_m &= \int_0^T s_m^2(t)dt = \sum_{n=1}^{N}\sum_{k=1}^{N} s_{m,n}s_{m,k}\int_0^T \psi_n(t)\psi_k(t)dt \\
&= \|\boldsymbol{s}_m\|^2 \\
< r, s_m > &= \int_0^T r(t)s_m(t)dt = \sum_{n=1}^{\infty}\sum_{k=1}^{N} r_n s_{m,k}\int_0^T \psi_n(t)\psi_k(t)dt \\
&= \boldsymbol{s}_m^T\boldsymbol{r}
\end{aligned}
$$

thus

$$
\hat{u} = \arg\max_m \left\{ \frac{\eta_0}{2}\log(p_m) - \frac{\mathcal{E}_m}{2} + < r, \boldsymbol{s}_m > \right\}
$$

MAP Receiver Architecture (2/2)



ML decision

$$
\begin{aligned}
\hat{u} &= \arg\max_m \left\{ f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) \right\} \\
&= \arg\max_m \left\{ \frac{1}{(\pi\eta_0)^{N/2}} \exp\left(-\frac{1}{\eta_0}\|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right) \right\} \\
&= \arg\min_m \left\{ \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right\}
\end{aligned}
$$

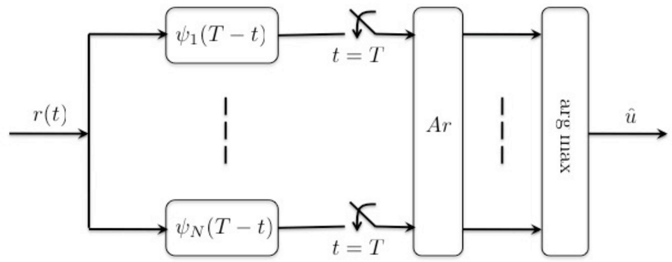
i.e. minimum distance decision rule

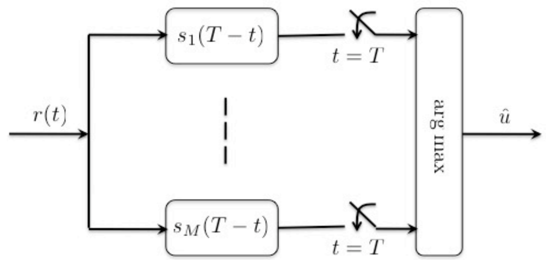
$$
\begin{aligned}
\hat{u} &= \arg\max_m \left\{ \boldsymbol{s}_m^T\boldsymbol{r} - \frac{\|\boldsymbol{s}_m\|^2}{2} \right\} \\
&= \arg\max_m \left\{ < r, \boldsymbol{s}_m > - \frac{\mathcal{E}_m}{2} \right\}
\end{aligned}
$$

If signals have equal energy $\mathcal{E}_m = \mathcal{E}$ the decision is

$$
\hat{u} = \arg\max_m \left\{ \boldsymbol{s}_m^T\boldsymbol{r} \right\} = \arg\max_m \left\{ < r, \boldsymbol{s}_m > \right\}
$$

ML Receiver Architecture with equal-energy signals





Error Probability

$$\begin{aligned} P_e &= 1 - P_c = 1 - \sum_{m=1}^{M} p_m \Pr(c|H_m) \\ &= 1 - \sum_{m=1}^{M} p_m \Pr\left(\boldsymbol{r}|H_m \in \Omega_m\right) \\ &= 1 - \sum_{m=1}^{M} p_m \int_{\Omega_m} f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) d\boldsymbol{r} \\ &= 1 - \sum_{m=1}^{M} p_m \Pr\left(\mathcal{N}\left(\boldsymbol{s}_m, \frac{\eta_0}{2}\boldsymbol{I}_N\right) \in \Omega_m\right) \\ &= 1 - \sum_{m=1}^{M} \frac{p_m}{(\pi\eta_0)^{N/2}} \int_{\Omega_m} \exp\left(-\frac{1}{\eta_0}\|\boldsymbol{r} - \boldsymbol{s}_m\|^2\right) d\boldsymbol{r} \end{aligned}$$

Signal-to-Noise Ratio

Symbol Error Rate (SER) takes into account for the average number of symbols in error w.r.t. the number of transmitted symbols.
Bit Error Rate (BER) takes into account for the average number of bits in error w.r.t. the number of transmitted bits (we will come on this later).
SER and BER are usually expressed as function of two possible Signal-to-Noise ratios (SNRs): SNR per symbol (γ) or SNR per bit (γ_b)

$$\begin{aligned} \gamma &= \mathcal{E}/\eta_0 \\ \gamma_b &= \mathcal{E}_b/\eta_0 \end{aligned}$$

where

$$\mathcal{E} = \sum_{m=1}^{M} p_m \mathcal{E}_m \qquad \mathcal{E}_b = \frac{\mathcal{E}}{\log_2(M)}$$

are the average energy per symbol and the average energy per bit