Digital Communications

— Lecture 05 —
Popular Modulation Formats

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Outline

1. ASK
2. PSK, BPSK, QPSK
3. ASK/PSK, QAM
4. Orthogonal Modulations, FSK

ASK - definition

Amplitude Shift Keying (ASK) modulation, often denoted discrete Pulse Amplitude Modulation (PAM)

Different signals are obtained changing the amplitude of one single pulse

$$s_m(t) = A_m\, p(t) \qquad m = 1, \dots, M$$

Amplitudes are usually assumed to be equally spaced and symmetric around the origin

$$A_m = -\frac{M-1}{2}\Delta + (m-1)\Delta \qquad m = 1, \dots, M$$

where Δ is the absolute difference between adjacent amplitudes

ASK - constellation

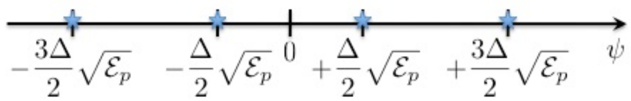
The signal space has dimension $N = 1$, thus

$$\psi(t) = \frac{1}{\sqrt{\mathcal{E}_p}}p(t) \qquad \text{where} \qquad \mathcal{E}_p = \int_0^T p^2(t)dt$$

The single component representing the mth signal is

$$s_m = A_m\sqrt{\mathcal{E}_p}$$

thus the signal constellation is ($M = 4$)



ASK - distance set

The distance set from the mth constellation point is dependent on m

The constellation does not exhibit a symmetric scenario w.r.t. each signal

The minimum distance is $d_{min} = \Delta\sqrt{\mathcal{E}_p}$

Each outer constellation point has 1 neighbor at minimum distance
Each inner constellation point has 2 neighbors at minimum distance

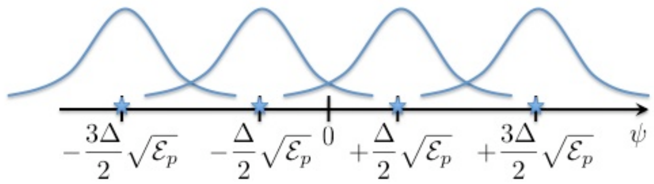
ASK - received signal

M possible hypotheses in $t \in [0, T)$

$$H_m \quad : \quad r(t) = A_m\, p(t) + w(t) \qquad m = 1, \ldots, M$$

The sufficient statistic is

$$r|H_m = A_m\sqrt{\mathcal{E}_p} + w \qquad \sim \mathcal{N}\left(A_m\sqrt{\mathcal{E}_p}, \eta_0/2\right)$$

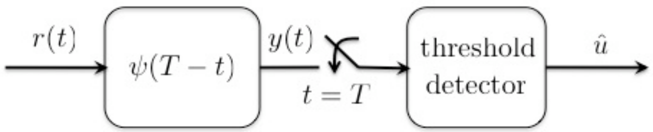


ASK - MAP (1/2)

The MAP decision rule is

$$\begin{aligned}\hat{u} &= \arg\max_m \left\{p_m f_{r|H_m}(r)\right\} \\ &= \arg\max_m \left\{\eta_0 \log(p_m) - (r - s_m)^2\right\} \\ &= \arg\max_m \left\{2 s_m r + \eta_0 \log(p_m) - s_m^2\right\}\end{aligned}$$

and corresponds to the following receiver



ASK - MAP (2/2)

The thresholds $\{\lambda_0, \ldots, \lambda_M\}$ are defined as follows

$$r \in (\lambda_{m-1}, \lambda_m) \qquad \rightarrow \qquad \hat{u} = a_m$$

thus $\lambda_0 = -\infty$, $\lambda_M = +\infty$, while for the inner thresholds

$$p_m f_{r|H_m}(\lambda_m) = p_{m+1} f_{r|H_{m+1}}(\lambda_m)$$

i.e.

$$\lambda_m = \frac{s_{m+1} + s_m}{2} + \frac{\eta_0}{2(s_{m+1} - s_m)} \log\left(\frac{p_m}{p_{m+1}}\right) \qquad m = 1, \ldots, M-1$$

For ML decision rule

$$\lambda_m = \frac{s_{m+1} + s_m}{2} \qquad m = 1, \ldots, M-1$$

ASK - SER (1/3)

SER for equiprobable symbols with ML detection is

$$
\begin{aligned}
P_e &= \frac{1}{M}\sum_{m=1}^{M}\Pr(e|H_m) \\
&= \frac{1}{M}\left(2\Pr(e|H_1) + (M-2)\Pr(e|H_2)\right) \\
&= \frac{1}{M}\left(2Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right) + (M-2)2Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)\right) \\
&= \frac{2(M-1)}{M}Q\left(\Delta\sqrt{\frac{\mathcal{E}_p}{2\eta_0}}\right)
\end{aligned}
$$

ASK - SER (2/3)

Average energy per symbol, in case of equiprobable symbols, is

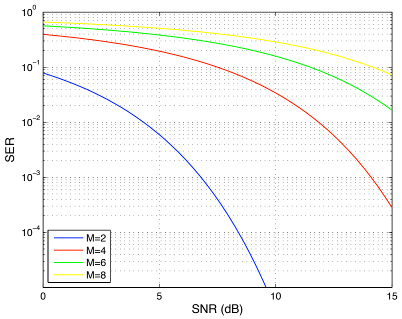
$$\mathcal{E} = \frac{1}{M}\sum_{m=1}^{M} A_m^2\mathcal{E}_p = \frac{M^2-1}{12}\Delta^2\mathcal{E}_p$$

where $\sum_{m=1}^{M} m = M(M+1)/2$ and $\sum_{m=1}^{M} m^2 = M(M+1)(2M+1)/6$ have been used, thus

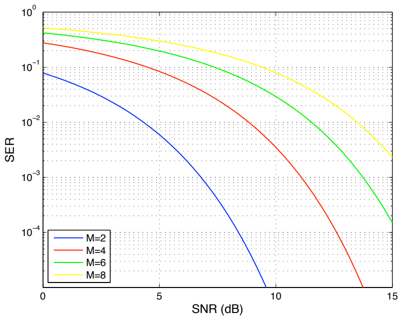
$$
\begin{aligned}
P_e &= 2\left(1-\frac{1}{M}\right)Q\left(\sqrt{\frac{6}{M^2-1}\gamma}\right) \\
&= 2\left(1-\frac{1}{M}\right)Q\left(\sqrt{\frac{6\log_2(M)}{M^2-1}\gamma_b}\right)
\end{aligned}
$$

ASK - SER (3/3)

$P_e(\gamma)$



$P_e(\gamma_b)$



The energy required for a target P_e increases with the cardinality M

PSK - definition

Phase Shift Keying modulation

Different signals are obtained changing the phase of one single sinusoid

$$s_m(t) = A\cos(2\pi f_0 t + \theta_m)\mathrm{rect}\left(\frac{t-T/2}{T}\right) \qquad m = 1,\ldots,M$$

Phases are usually assumed to be equally spaced in $[0, 2\pi)$ or $[-\pi, +\pi)$

$$\theta_m = \phi + (m-1)\frac{2\pi}{M} \qquad m = 1,\ldots,M$$

where ϕ is the initial phase reference (often assumed null)

One of the following conditions is usually assumed

$$2f_0T \in \mathbb{N}$$
$$f_0T >> 1$$

PSK - constellation (1/2)

Signal have all the same energy

$$\begin{aligned}\mathcal{E} &= \int_0^T A^2 \cos^2(2\pi f_0 t + \theta_m) dt \\ &= \frac{A^2 T}{2} + \frac{A^2}{2} \int_0^T \cos(2\pi 2 f_0 t + 2\theta_m) dt \\ &\approx \frac{A^2 T}{2}\end{aligned}$$

The signal space has dimension $N = 2$ as

$$s_m(t) = (A\cos(\theta_m)\cos(2\pi f_0 t) - A\sin(\theta_m)\sin(2\pi f_0 t))\,\mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

and

$$\int_0^T \cos(2\pi f_0 t)\sin(2\pi f_0 t) dt \approx 0$$

PSK - constellation (2/2)

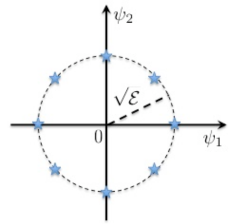
A possible orthonormal set of signals is

$$\begin{aligned}\psi_1(t) &= +\sqrt{\frac{2}{T}}\cos(2\pi f_0 t)\mathrm{rect}\left(\frac{t - T/2}{T}\right) \\ \psi_2(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_0 t)\mathrm{rect}\left(\frac{t - T/2}{T}\right)\end{aligned}$$

The (2-dimensional) vector representing the mth signal is

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ s_{m,2} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ < s_m, \psi_2 > \end{pmatrix} = \begin{pmatrix} \sqrt{\mathcal{E}}\cos(\theta_m) \\ \sqrt{\mathcal{E}}\sin(\theta_m) \end{pmatrix}$$

thus the signal constellation is
($M = 8$ and $\phi = 0$)



PSK - distance set

The distance set from the mth constellation point is NOT dependent on m

The constellation DOES exhibit a symmetric scenario w.r.t. each signal

The minimum distance is $d_{min} = 2\sqrt{\mathcal{E}}\sin(\pi/M)$

Each constellation point has 2 neighbors at minimum distance

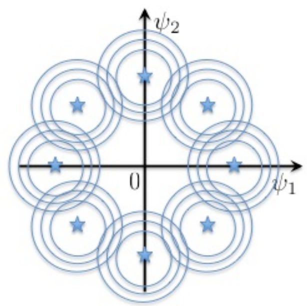
PSK - received signal

M possible hypotheses in $t \in [0, T)$

$$H_m \quad : \quad r(t) = A\cos(2\pi f_0 t + \theta_m) + w(t) \qquad m = 1, \ldots, M$$

The sufficient statistic is

$$\boldsymbol{r}|H_m = \boldsymbol{s}_m + \boldsymbol{w} \qquad \sim \mathcal{N}\left(\boldsymbol{s}_m, \frac{\eta_0}{2}\boldsymbol{I}_2\right)$$

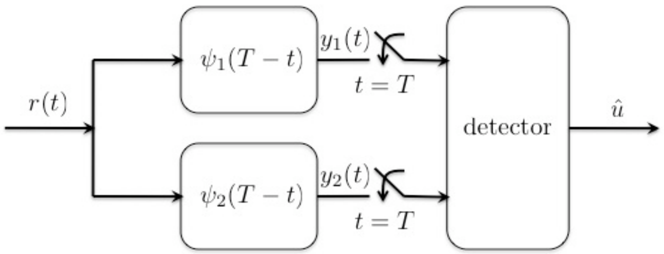


PSK - MAP

The MAP decision rule is

$$\begin{aligned}\hat{u} &= \arg\max_m \{p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r})\} \\ &= \arg\max_m \{\eta_0 \log(p_m) - \|\boldsymbol{r} - \boldsymbol{s}_m\|^2\} \\ &= \arg\max_m \left\{ r_1 \cos(\theta_m) + r_2 \sin(\theta_m) + \frac{\eta_0}{2\sqrt{\mathcal{E}}} \log(p_m) \right\}\end{aligned}$$

and corresponds to the following receiver

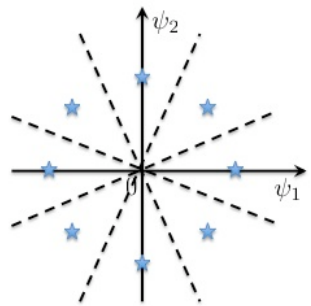


PSK - ML

The ML decision rule is

$$\begin{aligned}\hat{u} &= \arg\min_m \{\|\boldsymbol{r} - \boldsymbol{s}_m\|^2\} \\ &= \arg\max_m \{r_1 \cos(\theta_m) + r_2 \sin(\theta_m)\}\end{aligned}$$

and corresponds to the following regions



PSK - SER (1/2)

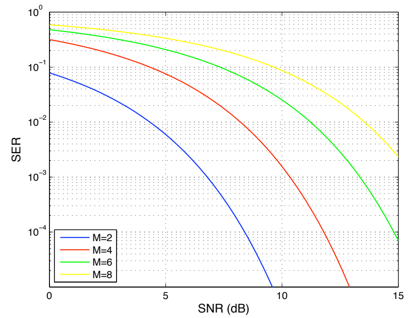
SER for equiprobable symbols with ML detection is

$$\begin{aligned}P_e &= 1 - \frac{1}{M} \sum_{m=1}^{M} \Pr(c|H_m) = 1 - \Pr(c|H_1) \\ &= 1 - \iint_{\Omega_1} \frac{1}{\pi\eta_0} \exp\left(- \frac{\left(r_1 - \sqrt{\mathcal{E}}\right)^2 + r_2^2}{\eta_0} \right) dr_1 dr_2 \\ &= 1 - \frac{\exp(-\mathcal{E}/\eta_0)}{\pi\eta_0} \int_{-\pi/M}^{+\pi/M} \int_0^{+\infty} \rho \exp\left(\frac{2\sqrt{\mathcal{E}}\rho\cos(\theta) - \rho^2}{\eta_0} \right) d\rho d\theta\end{aligned}$$

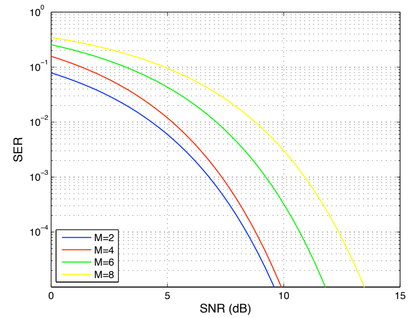
Average energy per symbol is $\mathcal{E}$ and average energy per bit is $\mathcal{E}/\log_2(M)$

PSK - SER (2/2)

$P_e(\gamma)$



$P_e(\gamma_b)$



The energy required for a target P_e increases with the cardinality M

BPSK and QPSK

Binary Phase Shift Keying (BPSK) is a binary modulation format with signals $\{-p(t), +p(t)\}$, thus including both ASK and PSK with $M = 2$

$$\begin{aligned} P_e &= Q\left(\sqrt{2\gamma}\right) \\ &= Q\left(\sqrt{2\gamma_b}\right) \end{aligned}$$

Quaternary Phase Shift Keying (QPSK) is a quaternary modulation format with signals $\{-p_1(t), +p_1(t), -p_2(t), +p_2(t)\}$, with $< p_1, p_2 >= 0$, thus including PSK with $M = 4$

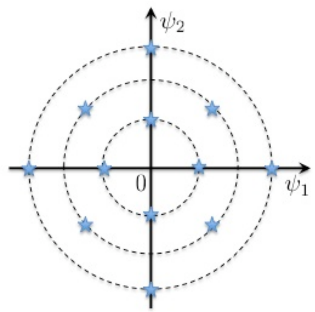
$$\begin{aligned} P_e = 2Q\left(\sqrt{\gamma}\right) - Q^2\left(\sqrt{\gamma}\right) &\approx 2Q\left(\sqrt{\gamma}\right) \\ &= 2Q\left(\sqrt{2\gamma_b}\right) \end{aligned}$$

ASK/PSK

It combines both ASK and PSK modulations, thus different signals are obtained changing the amplitude and/or the phase of a sinusoid

$$s_m(t) = A_m \cos(2\pi f_0 t + \theta_m)\mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

The signal space has dimension $N = 2$
There is no general expression for the performance including all cases

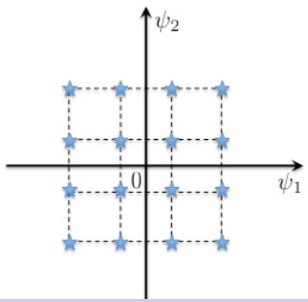


QAM - definition

Quadrature Amplitude Modulation (QAM) is the most common format in current communication systems.
It is a special case of ASK/PSK modulation with signal constellation placed on a regular grid.

$$s_m(t) = (A_m \cos(2\pi f_0 t) + B_m \sin(2\pi f_0 t))\,\mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

The signal space has dimension $N = 2$. Usually $M = 2^{2p}$.



QAM - distance set

The distance set from the mth constellation point is dependent on m

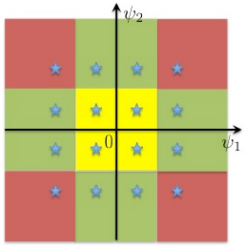
The constellation does not exhibit a symmetric scenario w.r.t. each signal

The minimum distance is $d_{min} = \Delta\sqrt{\mathcal{E}_p}$

Each corner constellation point has 2 neighbors at minimum distance
Each outer constellation point has 3 neighbors at minimum distance
Each inner constellation point has 4 neighbors at minimum distance

QAM - ML

Assuming $M = 2^{2p}$, three kinds of integrals must be computed and combined appropriately



$$
\begin{aligned}
P_{\text{red}} &= \left(1 - Q\left(\frac{d}{\sqrt{2\eta_0}}\right)\right)^2 \approx 1 - 2Q\left(\frac{d}{\sqrt{2\eta_0}}\right) \\
P_{\text{green}} &= \left(1 - Q\left(\frac{d}{\sqrt{2\eta_0}}\right)\right)\left(1 - 2Q\left(\frac{d}{\sqrt{2\eta_0}}\right)\right) \\
&\approx 1 - 3Q\left(\frac{d}{\sqrt{2\eta_0}}\right) \\
P_{\text{yellow}} &= \left(1 - 2Q\left(\frac{d}{\sqrt{2\eta_0}}\right)\right)^2 \approx 1 - 4Q\left(\frac{d}{\sqrt{2\eta_0}}\right)
\end{aligned}
$$

$$
\begin{aligned}
P_e &= 1 - \frac{4}{M}\left(P_{\text{red}} + (\sqrt{M} - 2)P_{\text{green}} + (M/4 + 1 - \sqrt{M})P_{\text{yellow}}\right) \\
&\approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\frac{d}{\sqrt{2\eta_0}}\right)
\end{aligned}
$$

QAM - SER (1/2)

Assuming $M = 2^{2p}$, the average energy of the constellation is twice the average energy of a PAM constellation with cardinality $\sqrt{M}$, thus

$$\mathcal{E} = \frac{M-1}{6} d^2$$

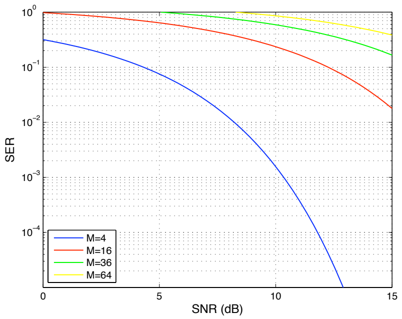
and

$$
\begin{aligned}
P_e &\approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\gamma}\right) \\
&= 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2(M)}{M-1}\gamma_b}\right)
\end{aligned}
$$

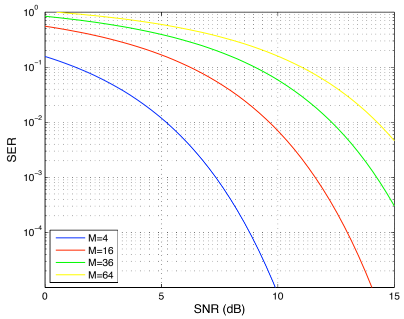
from which it is worth noticing that $\text{SER}(\gamma_b)$ of $M^2 - \text{QAM}$ is twice $\text{SER}(\gamma_b)$ of $M - \text{PAM}$ with equal distance among adjacent symbols

QAM - SER (2/2)

$P_e(\gamma)$



$P_e(\gamma_b)$



The energy required for a target P_e increases with the cardinality M

Orthogonal Modulation - definition

The set of signals is made of M orthogonal waveforms, i.e.

$$< s_m, s_\ell >= \mathcal{E}_m \delta_{m,\ell}$$

A possible orthonormal set of signals is simply obtained as

$$\psi_m(t) = \frac{1}{\sqrt{\mathcal{E}_m}} s_m(t) \qquad m = 1, \dots, M$$

The signal space has dimension $N = M$

Signals are usually chosen with equal energy, thus

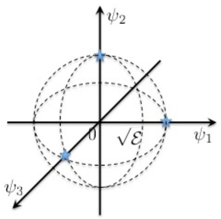
$$\mathcal{E}_m = \mathcal{E} \qquad m = 1, \dots, M$$

Orthogonal Modulation - constellation

The (M-dimensional) vector representing the mth signal is

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,m-1} \\ s_{m,m} \\ s_{m,m+1} \\ \vdots \\ s_{m,M} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_{m-1} > \\ < s_m, \psi_m > \\ < s_m, \psi_{m+1} > \\ \vdots \\ < s_m, \psi_M > \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{\mathcal{E}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

thus the signal constellation is ($M = 3$)



Orthogonal Modulation - distance set

The distance set from the mth constellation point is NOT dependent on m

The constellation DOES exhibit a symmetric scenario w.r.t. each signal

The minimum distance is $d_{min} = \sqrt{2\mathcal{E}}$

Each constellation point has $M - 1$ neighbors at minimum distance

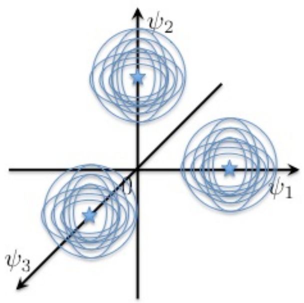
Orthogonal Modulation - received signal

M possible hypotheses in $t \in [0, T)$

$$H_m \quad : \quad r(t) = s_m(t) + w(t) \qquad m = 1, \ldots, M$$

The sufficient statistic is

$$\boldsymbol{r}|H_m = \boldsymbol{s}_m + \boldsymbol{w} \qquad \sim \mathcal{N}\left(\boldsymbol{s}_m, \frac{\eta_0}{2}\boldsymbol{I}_2\right)$$



Orthogonal Modulation - MAP and ML

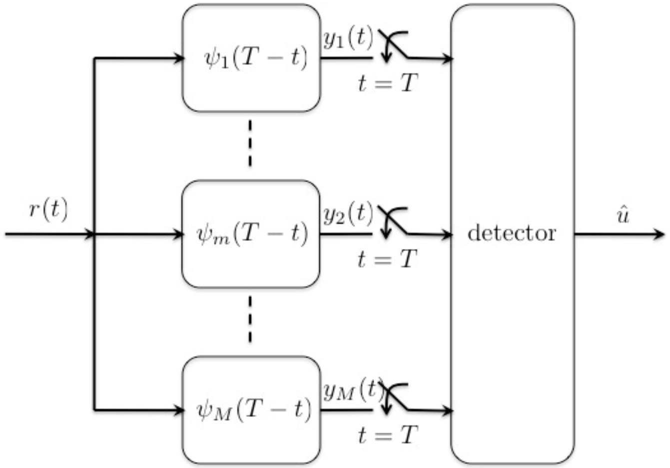
The MAP decision rule is

$$\begin{aligned} \hat{u} &= \arg\max_m \left\{ p_m f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) \right\} \\ &= \arg\max_m \left\{ \eta_0 \log(p_m) - \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right\} \\ &= \arg\max_m \left\{ r_m + \frac{\eta_0}{2\sqrt{\mathcal{E}_m}} \log(p_m) - \frac{\sqrt{\mathcal{E}_m}}{2} \right\} \end{aligned}$$

The ML decision rule, in case of signals with equal energy, is

$$\begin{aligned} \hat{u} &= \arg\min_m \left\{ \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right\} \\ &= \arg\max_m \left\{ r_m \right\} \end{aligned}$$

Orthogonal Modulation - receiver



Orthogonal Modulation - SER (1/2)

SER for equiprobable and equal-energy symbols with ML detection is

$$P_e = 1 - \frac{1}{M}\sum_{m=1}^{M} \Pr(c|H_m) = 1 - \Pr(c|H_1)$$

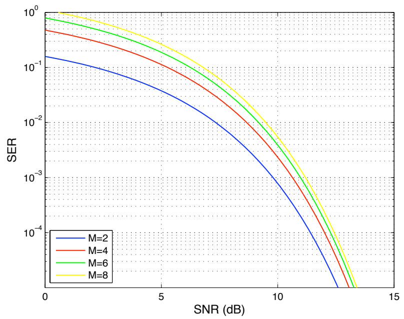
$$= 1 - \int \cdots \int_{\Omega_1} \left(\frac{1}{\pi\eta_0}\right)^{M/2} \exp\left(-\frac{\left(r_1 - \sqrt{\mathcal{E}}\right)^2 + \sum_{m=2}^{M} r_m^2}{\eta_0}\right) dr_1 \ldots dr_M$$

$$= 1 - \frac{1}{\sqrt{\pi\eta_0}} \int_{\mathbb{R}} \exp\left(-\frac{(r-\sqrt{\mathcal{E}})^2}{\eta_0}\right)\left(1 - Q\left(\frac{r}{\sqrt{\eta_0/2}}\right)\right)^{M-1} dr$$

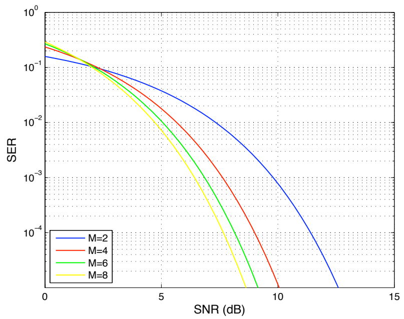
Average energy per symbol is $\mathcal{E}$ and average energy per bit is $\mathcal{E}/\log_2(M)$

Orthogonal Modulation - SER (2/2)

$P_e(\gamma)$



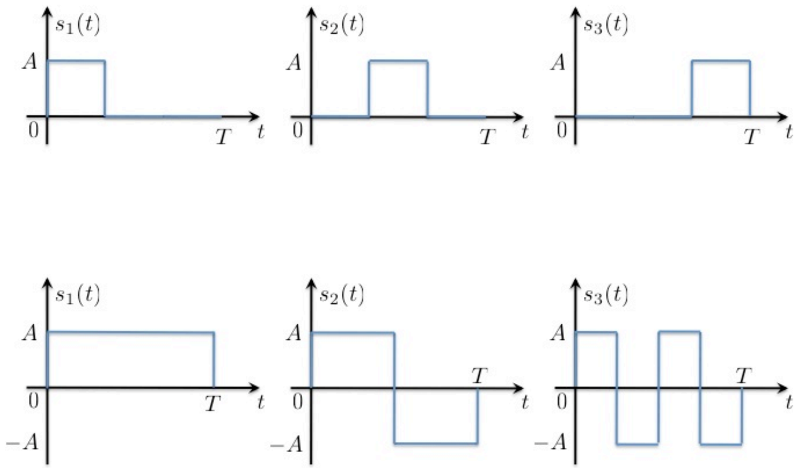
$P_e(\gamma_b)$



The energy required for a target P_e decreases with the cardinality M

The limit is $\gamma_b > \log(2) = -1.6$ dB

Orthogonal Modulation - examples



FSK - definition

Frequency Shift Keying (ASK) modulation, different signals are obtained changing the frequency of one single sinusoid

$$s_m(t) = A\cos(2\pi f_m t + \phi_m)\text{rect}\left(\frac{t - T/2}{T}\right) \qquad m = 1, \ldots, M$$

where phases are usually assumed to be equal ($\phi_m = 0$)

Frequencies are usually assumed to be equally spaced and symmetric around the carrier frequency (f_0)

$$f_m = f_0 - \frac{M-1}{2}\Delta + (m-1)\Delta \qquad m = 1, \ldots, M$$

where Δ is the absolute difference between adjacent frequencies

Also, $f_0 T >> 1$ is usually assumed

FSK - orthogonality

$$
\begin{aligned}
< s_m, s_\ell >= & \int_0^T A^2 \cos(2\pi f_m t + \phi_m)\cos(2\pi f_\ell t + \phi_\ell)dt \\
\approx & \frac{A^2}{2}\int_0^T \cos(2\pi(m-\ell)\Delta t + \phi_m - \phi_\ell)dt \\
= & \frac{A^2}{4\pi(m-\ell)\Delta}\left(\sin(2\pi(m-\ell)\Delta T + \phi_m - \phi_\ell) - \sin(\phi_m - \phi_\ell)\right)
\end{aligned}
$$

In order to have orthogonal FSK, i.e. $< s_m, s_\ell >= \mathcal{E}\delta_{m,\ell} = (A^2T/2)\delta_{m,\ell}$, the following condition is needed

$$\Delta = \frac{k}{T} \qquad \Delta_{min} = \frac{1}{T}$$

while in case of equal phases (e.g. $\phi_m = 0$)

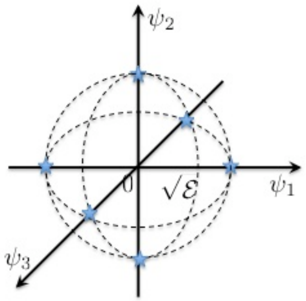
$$\Delta = \frac{k}{2T} \qquad \Delta_{min} = \frac{1}{2T}$$

Biorthogonal Signals - definition

A set of M biorthogonal signals is made of $M/2$ orthogonal signals plus their $M/2$ negatives

$$< s_m, s_\ell >= \begin{cases} +\mathcal{E} & m = \ell \\ -\mathcal{E} & |m - \ell| = M/2 \\ 0 & \text{else} \end{cases}$$

The signal space has dimension $N = M/2$



Biorthogonal Modulation - constellation

The ($M/2$-dimensional) vector representing the mth signal is

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,m} \\ \vdots \\ s_{m,M/2} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_m > \\ \vdots \\ < s_m, \psi_{M/2} > \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ +\sqrt{\mathcal{E}} \\ \vdots \\ 0 \end{pmatrix}$$

The ($M/2$-dimensional) vector representing the $(m + M/2)$th signal is

$$\boldsymbol{s}_{m+M/2} = \begin{pmatrix} s_{m+M/2,1} \\ \vdots \\ s_{m+M/2,m} \\ \vdots \\ s_{m+M/2,M/2} \end{pmatrix} = \begin{pmatrix} < s_{m+M/2}, \psi_1 > \\ \vdots \\ < s_{m+M/2}, \psi_m > \\ \vdots \\ < s_{m+M/2}, \psi_{M/2} > \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ -\sqrt{\mathcal{E}} \\ \vdots \\ 0 \end{pmatrix}$$

Biorthogonal Modulation - distance set

The distance set from the mth constellation point is NOT dependent on m

The constellation DOES exhibit a symmetric scenario w.r.t. each signal

The minimum distance is $d_{min} = \sqrt{2\mathcal{E}}$

Each constellation point has $M-2$ neighbors at minimum distance and also 1 neighbor at distance $2\sqrt{\mathcal{E}}$

Biorthogonal Modulation - ML

The ML decision rule, in case of signals with equal energy, is

$$\begin{aligned}\hat{u} &= \arg\min_m \{\|\boldsymbol{r} - \boldsymbol{s}_m\|^2\} \\ &= \text{sign}\left(\arg\max_n \{|r_n|\}\right)\end{aligned}$$

i.e.

$$\begin{aligned}\Omega_1 &= \{r_1 > |r_n| \ \forall n \neq 1,\ r_1 > 0\} \\ &\dots \\ \Omega_{M/2} &= \{r_{M/2} > |r_n| \ \forall n \neq M/2,\ r_{M/2} > 0\} \\ \Omega_{1+M/2} &= \{r_1 > |r_n| \ \forall n \neq 1,\ r_1 < 0\} \\ &\dots \\ \Omega_M &= \{r_{M/2} > |r_n| \ \forall n \neq M/2,\ r_{M/2} < 0\}\end{aligned}$$

Biorthogonal Modulation - SER

SER for equiprobable and equal-energy symbols with ML detection is

$$P_e = 1 - \frac{1}{M}\sum_{m=1}^{M} \Pr(c|H_m) = 1 - \Pr(c|H_1)$$

$$= 1 - \int \cdots \int_{\Omega_1} \left(\frac{1}{\pi\eta_0}\right)^{M/4} \exp\left(-\frac{\left(r_1 - \sqrt{\mathcal{E}}\right)^2 + \sum_{m=2}^{M/2} r_m^2}{\eta_0}\right) dr_1 \dots dr_{\frac{M}{2}}$$

$$= 1 - \frac{1}{\sqrt{\pi\eta_0}} \int_0^{+\infty} \exp\left(-\frac{(r-\sqrt{\mathcal{E}})^2}{\eta_0}\right)\left(1 - 2Q\left(\frac{r}{\sqrt{\eta_0/2}}\right)\right)^{\frac{M}{2}-1} dr$$

Average energy per symbol is $\mathcal{E}$ and average energy per bit is $\mathcal{E}/\log_2(M)$

Performance are similar to orthogonal modulation

Simplex Signals - definition

A set of M simplex signals is built upon a set of M orthogonal signals as

$$\tilde{s}_m(t) = s_m(t) - s_\mu(t) \qquad m = 1, \dots, M$$

where $\{s_1(t), \dots, s_M(t)\}$ is the set of orthogonal signals and where $s_\mu(t) = (1/M)\sum_{m=1}^{M} s_m(t)$ is the mean signal

Shift of the origin in the signal space, thus distances are kept

Simplex signals are equally correlated

$$< s_m, s_\ell >= \begin{cases} \left(\frac{M-1}{M}\right)\mathcal{E}_o = \mathcal{E} & m = \ell \\ -\frac{1}{M}\mathcal{E}_o & m \neq \ell \end{cases}$$

The signal space has dimension $N = M - 1$ $(\sum_{m=1}^{M} \tilde{s}_m(t) = 0)$

Simplex Signals - constellation

The (M-dimensional) vector representing the mth signal (in the original signal space) is

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,m-1} \\ s_{m,m} \\ s_{m,m+1} \\ \vdots \\ s_{m,M} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_{m-1} > \\ < s_m, \psi_m > \\ < s_m, \psi_{m+1} > \\ \vdots \\ < s_m, \psi_M > \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{\mathcal{E}_o}}{M} \\ \vdots \\ -\frac{\sqrt{\mathcal{E}_o}}{M} \\ (\frac{M-1}{M})\sqrt{\mathcal{E}_o} \\ -\frac{\sqrt{\mathcal{E}_o}}{M} \\ \vdots \\ -\frac{\sqrt{\mathcal{E}_o}}{M} \end{pmatrix}$$

Simplex Signals - SER

Distance-set properties are the same as for the original orthogonal set

SER is the same as an orthogonal modulation with $\mathcal{E}_o = \left(\frac{M}{M-1}\right)\mathcal{E}$

$$P_e = 1 - \int_{\mathbb{R}} \exp\left(-\frac{\left(r - \sqrt{\left(\frac{M}{M-1}\right)\mathcal{E}}\right)^2}{\eta_0}\right)\left(1 - Q\left(\frac{r}{\sqrt{\eta_0/2}}\right)\right)^{M-1} \frac{dr}{\sqrt{\pi\eta_0}}$$

i.e. a gain of $10\log_{10}\left(\frac{M}{M-1}\right)$ dB

Signals from Binary Codes

Consider a set of N orthogonal signals, e.g.

$$p_n(t) = \sqrt{\frac{2\mathcal{E}_c}{T_c}}\cos(2\pi f_c t)\mathrm{rect}\left(\frac{t - T_c/2 - nT_c}{Tc}\right) \quad , \quad n = 1, \dots, N$$

with $T_c = T/N$ and $\mathcal{E}_c = \mathcal{E}/N$

Consider a binary representation of the $M \leq 2^N$ symbols,
$a_m \equiv (b_{m,1}, \dots, b_{m,N})$ with $b_{m,n} \in \{0, 1\}$
BPSK can be used to represent the nth component of the codeword

$$b_{m,n} = 0 \rightarrow -p_n(t) \qquad , \qquad b_{m,n} = 1 \rightarrow +p_n(t)$$

thus $c_{m,n} = 2b_{m,n} - 1 \in \{-1, +1\}$ and

$$s_m(t) = \sum_{n=1}^{N} c_{m,n} p_n(t), \qquad m = 1, \dots, M$$

Signals from Binary Codes - constellation

The (N-dimensional) vector representing the mth signal

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,m} \\ \vdots \\ s_{m,M} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_m > \\ \vdots \\ < s_m, \psi_M > \end{pmatrix} = \begin{pmatrix} c_{m,1}\sqrt{\frac{\mathcal{E}}{N}} \\ \vdots \\ c_{m,m}\sqrt{\frac{\mathcal{E}}{N}} \\ \vdots \\ c_{m,N}\sqrt{\frac{\mathcal{E}}{N}} \end{pmatrix}$$

i.e. one of the 2^N vertices of an N-dimensional hypercube