Digital Communications

— Lecture 06 —
Bounds on SER and BER

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Outline

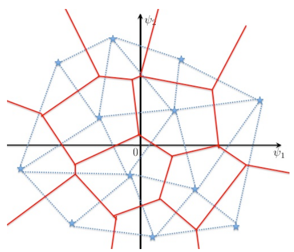
ML Receiver and Voronoi Diagram

- $\mathcal{A} = \{a_1, \ldots, a_M\}$: M-ary modulation
- $\Pi = \{1/M, \ldots, 1/M\}$: equiprobable symbols
- $\{s_1(t), \ldots, s_M(t)\},\ t \in [0, T)$: arbitrary modulation format
- $\{\psi_1(t), \ldots, \psi_N(t)\},\ t \in [0, T)$: N-dimensional signal space
- $\boldsymbol{s}_m = (s_{m,1}, \ldots, s_{m,N})^T \in \mathbb{R}^N$: mth constellation point

The ML detector corresponds to the Voronoi diagram around each constellation point, bounded by hyperplanes in $\mathbb{R}^N$

Given M reference points in $\mathbb{R}^N$, the Voronoi diagram is the partition of $\mathbb{R}^N$ into M regions, namely the Voronoi cells, each including the points closer to a specific reference point than others

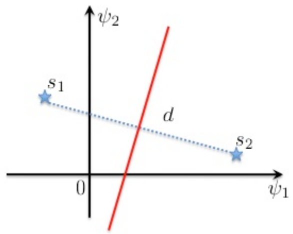
Exact SER and Voronoi Cells



The mth decision region is the mth Voronoi cell
$\Omega_m = \{\boldsymbol{r} \in \mathbb{R}^N : \|\boldsymbol{r} - \boldsymbol{s}_m\| \leq \|\boldsymbol{r} - \boldsymbol{s}_\ell\| \ \forall \ell \neq m\}$

$$
\begin{aligned}
P_e &= 1 - \frac{1}{M}\sum_{m=1}^{M} \Pr(c|H_m) = 1 - \frac{1}{M}\sum_{m=1}^{M} \Pr\left(\boldsymbol{r}|H_m \in \Omega_m\right) \\
&= 1 - \frac{1}{M}\sum_{m=1}^{M} \int_{\Omega_m} f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) d\boldsymbol{r} \\
&= 1 - \frac{1}{M}\sum_{m=1}^{M} \frac{1}{(\pi\eta_0)^{N/2}} \int_{\Omega_m} \exp\left(-\frac{1}{\eta_0}\|\boldsymbol{r} - \boldsymbol{s}_m\|^2\right) d\boldsymbol{r}
\end{aligned}
$$

SER-BER for Binary Modulation (1/3)



$$\boldsymbol{s}_1 = \begin{pmatrix} s_{1,1} \\ s_{1,2} \end{pmatrix} \quad , \quad \boldsymbol{s}_2 = \begin{pmatrix} s_{2,1} \\ s_{2,2} \end{pmatrix}$$

The correlation coefficient is

$$\rho \triangleq \frac{< \boldsymbol{s}_1, \boldsymbol{s}_2 >}{\sqrt{\mathcal{E}_1}\sqrt{\mathcal{E}_2}}$$

$$\begin{aligned} d^2 &= \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2 = \int_0^T (s_1(t) - s_2(t))^2 \, dt \\ &= \mathcal{E}_1 + \mathcal{E}_2 - 2\rho\sqrt{\mathcal{E}_1}\sqrt{\mathcal{E}_2} \end{aligned}$$

and in the case of equal-energy signals, i.e. $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$

$$d^2 = 2(1 - \rho)\mathcal{E}$$

SER-BER for Binary Modulation (2/3)

$$\begin{aligned} P_e &= 1 - \frac{1}{2}\Pr(c|H_1) - \frac{1}{2}\Pr(c|H_2) = 1 - \Pr(c|H_1) \\ &= 1 - \Pr\left(\mathcal{N}\left(\boldsymbol{s}_1, \frac{\eta_0}{2}\boldsymbol{I}_2\right) \in \Omega_1\right) \\ &= 1 - \frac{1}{\pi\eta_0}\iint_{\Omega_1} \exp\left(-\frac{(r_1 - s_{1,1})^2 + (r_2 - s_{1,2})^2}{\eta_0}\right) dr_1 dr_2 \end{aligned}$$

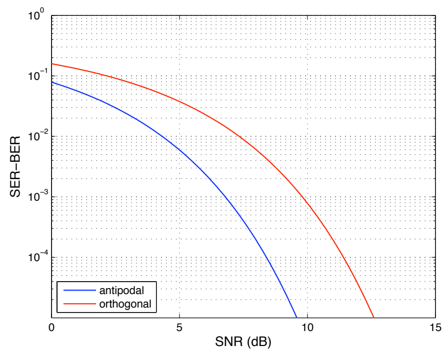
Integral computation is straightforward if we choose the blue dotted line and the red line as reference axis

$$\begin{aligned} P_e &= 1 - \Pr\left(\mathcal{N}\left(\begin{pmatrix} -d/2 \\ 0 \end{pmatrix}, \frac{\eta_0}{2}\boldsymbol{I}_2\right) \in \Omega_1\right) \\ &= 1 - \frac{1}{\pi\eta_0}\int_{-\infty}^{0} d\xi_1 \int_{\mathbb{R}} d\xi_2 \exp\left(-\frac{(\xi_1 + d/2)^2 + \xi_2^2}{\eta_0}\right) \\ &= Q\left(\frac{d}{\sqrt{2\eta_0}}\right) = Q\left(\sqrt{(1-\rho)\frac{\mathcal{E}}{\eta_0}}\right) \end{aligned}$$

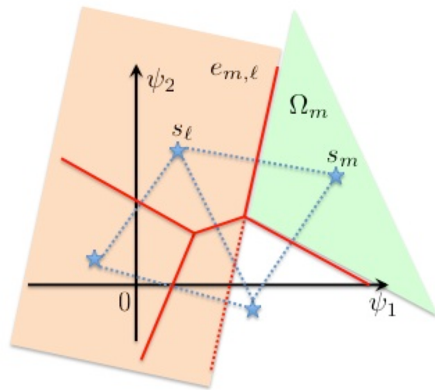
SER-BER for Binary Modulation (3/3)

$$P_e = Q\left(\sqrt{(1-\rho)\gamma}\right) = Q\left(\sqrt{(1-\rho)\gamma_b}\right)$$

- antipodal signals (i.e. $\rho = -1$): $P_e = Q\left(\sqrt{2\gamma}\right)$, best performance
- orthogonal signals (i.e. $\rho = 0$): $P_e = Q\left(\sqrt{\gamma}\right)$, 3 dB loss



Binary Error Event



$$e_{m,\ell} \triangleq \left\{\boldsymbol{r} \in \mathbb{R}^N : \|\boldsymbol{r} - \boldsymbol{s}_\ell\| \le \|\boldsymbol{r} - \boldsymbol{s}_m\| | H_m\right\}$$

i.e. the event that $\boldsymbol{r}|H_m$ is closer to $\boldsymbol{s}_\ell$ than $\boldsymbol{s}_m$

Upper Bound on SER

$e|H_m = \bigcup_{\ell=1,\ell\neq m}^{M} e_{m,\ell}$ and $e_{m,\ell_1} \bigcap e_{m,\ell_2} \neq \emptyset$, thus

$$\Pr(e|H_m) = \Pr\left(\bigcup_{\ell=1,\ell\neq m}^{M} e_{m,\ell}\right) \leq \sum_{\ell=1,\ell\neq m}^{M} \Pr\left(e_{m,\ell}\right) = \sum_{\ell=1,\ell\neq m}^{M} Q\left(\frac{d_{m,\ell}}{\sqrt{2\eta_0}}\right)$$

where $d_{m,\ell} = \|\boldsymbol{s}_m - \boldsymbol{s}_\ell\|$ is the (m,ℓ)th distance

$$P_e = \frac{1}{M}\sum_{m=1}^{M} \Pr(e|H_m) \leq \frac{1}{M}\sum_{m=1}^{M}\sum_{\ell=1,\ell\neq m}^{M} Q\left(\frac{d_{m,\ell}}{\sqrt{2\eta_0}}\right)$$

$Q(x)$ is decreasing with x, thus an upper bound is

$$P_e \leq (M-1)Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right)$$

where $d_{min} \triangleq \min_{m,\ell:m\neq\ell}\{d_{m,\ell}\}$ is the minimum distance

Lower Bound on SER (1/2)

Define $d_m \triangleq \min_{\ell:m\neq\ell}\{d_{m,\ell}\}$, i.e. the minimum distance from the mth constellation point, then

$$\Pr(e|H_m) \geq Q\left(\frac{d_m}{\sqrt{2\eta_0}}\right)$$

i.e. keep the binary error event providing the largest contribution to $e|H_m$

$$P_e = \frac{1}{M}\sum_{m=1}^{M} \Pr(e|H_m) \geq \frac{1}{M}\sum_{m=1}^{M} Q\left(\frac{d_m}{\sqrt{2\eta_0}}\right)$$

$Q(x)$ is $\cup-$concave with x, thus from Jensen's inequality

$$P_e \geq Q\left(\frac{d_\mu}{\sqrt{2\eta_0}}\right)$$

where $d_\mu \triangleq (1/M)\sum_{m=1}^{M} d_m$ is the average minimum distance

Lower Bound on SER (2/2)

Define $D_{min} \triangleq \{m \in \mathbb{N}_M : d_m = d_{min}\}$, i.e. the indices of constellation points with at least one neighbor at minimum distance, then

$$P_e \geq \frac{1}{M}\sum_{m=1}^{M} Q\left(\frac{d_m}{\sqrt{2\eta_0}}\right) \geq \frac{1}{M}\sum_{m\in D_{min}} Q\left(\frac{d_m}{\sqrt{2\eta_0}}\right) = \frac{M_{min}}{M} Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right)$$

where M_{min} is the cardinality of D_{min}

At least 2 constellation points are at minimum distance, thus $M_{min} \geq 2$

$$P_e \geq \frac{2}{M} Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right)$$

Approximate SER

The minimum distance is d_{min} is the parameter that summarizes the performance of a generic modulation format

Coherent with the fact that errors between close symbols are more frequent than errors between far-apart symbols

$$\frac{2}{M} Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right) \leq P_e \leq (M-1)Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right)$$

A popular approximation for SER is

$$P_e \approx M_{min} Q\left(\frac{d_{min}}{\sqrt{2\eta_0}}\right)$$

named the nearest-neighbor approximation

Exact BER

Bit Error Rate (BER) is usually preferred to SER because allows to compare modulation formats with different alphabet size

$$\begin{aligned} P_b &= \sum_{m=1}^{M}\sum_{\ell=1}^{M} \Pr(\text{bit in error}|H_m \to H_\ell)\Pr(H_m \to H_\ell) \\ &= \sum_{m=1}^{M}\sum_{\ell=1}^{M} \Pr(\text{bit in error}|H_m \to H_\ell)\Pr(\boldsymbol{s}_\ell|\boldsymbol{s}_m)p_m \\ &= \sum_{m=1}^{M}\sum_{\ell=1}^{M} \frac{E(\ell,m)}{\log_2(M)}\Pr(\boldsymbol{s}_\ell|\boldsymbol{s}_m)p_m \end{aligned}$$

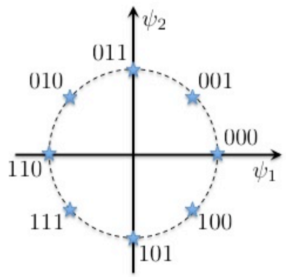
where

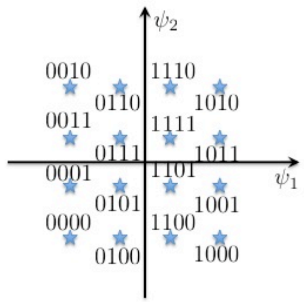
- $\Pr(\boldsymbol{s}_\ell|\boldsymbol{s}_m) = \int_{\Omega_\ell} f_{\boldsymbol{r}|H_m}(\boldsymbol{r})d\boldsymbol{r}$ is the probability of transmitting $\boldsymbol{s}_m$ and receiving $\boldsymbol{s}_\ell$ (depends on the signal constellation and the receiver)
- $E(\ell,m)$ is the number of bits in error when transmitting $\boldsymbol{s}_m$ and receiving $\boldsymbol{s}_\ell$ (depends on the labeling among symbols and bit strings)

Gray Labeling

Gray code is a binary numeral system such that two successive values differ in only one bit, i.e. adjacent symbols have Hamming distance equal to 1

ASK, PSK and QAM always allow Gray labeling





Approximate BER (1/2)

For ASK, PSK, QAM, or other modulations allowing for Gray labeling

- assume errors only occur between adjacent symbols (very likely at high SNR)
- each symbol in error provides 1 single bit in error over $\log_2(M)$ transmitted bits

$$P_b \approx \frac{1}{\log_2(M)} P_e$$

Approximate BER (2/2)

In case of orthogonal modulation (e.g. FSK)

- the constellation is totally symmetric, thus $P_e = \Pr(e|H_m) = (M-1)\Pr(\boldsymbol{s}_\ell|\boldsymbol{s}_m)$ with $m \neq \ell$
- each constellation point is at d_{min} from each other
- $M/2$ constellation points have the same the bit of interest as $\boldsymbol{s}_1$, and $M/2$ constellation points don't, thus $P_b = (M/2)\Pr(\boldsymbol{s}_\ell|\boldsymbol{s}_m)$ with $m \neq \ell$

$$\begin{aligned} P_b &= \frac{M/2}{M-1} P_e \\ &\approx \frac{1}{2} P_e \end{aligned}$$