Digital Communications

— Lecture 07 —
PSD of Digitally-Modulated Signals

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Outline

Digitally-Modulated Signal

- $\mathcal{A} = \{a_1, \dots, a_M\}$ is the alphabet
- $\{s_1(t), \dots, s_M(t)\}$ the set of signals ($t \in [0, T)$)

The signal associated to the message $\mathcal{U} = (\dots, u_{-1}, u_0, u_1, \dots, u_k, \dots)$ is

$$z(t, \mathcal{U}) = \sum_{k=-\infty}^{+\infty} s(t - kT; u_k)$$

where $s(t - kT; u_k = a_m) = s_m(t - kT)$

Auto-Correlation Function

$$R_z(t, \tau) = \sum_{k=-\infty}^{+\infty} \sum_{h=-\infty}^{+\infty} \mathbb{E}\{s(t - kT; u_k)s(t - \tau - hT; u_h)\}$$

The signal is ciclo-stationary, with period equal to the symbol time, if the sequence $\mathcal{U}$ is II-order stationary

$$R_z(t + \ell T, \tau) = \sum_{k=-\infty}^{+\infty} \sum_{h=-\infty}^{+\infty} \mathbb{E}\{s(t - (k - \ell)T; u_k)s(t - \tau - (h - \ell)T; u_h)\}$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{h=-\infty}^{+\infty} \mathbb{E}\{s(t - kT; u_{k+\ell})s(t - \tau - hT; u_{h+\ell})\} = R_z(t, \tau)$$

Mean Auto-Correlation

$$
\begin{aligned}
R_z(\tau) &= \frac{1}{T}\int_0^T R_z(t,\tau)dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty}\int_0^T \mathbb{E}\left\{s(t-kT;u_k)s(t-\tau-hT;u_h)\right\}dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\sum_{\ell=-\infty}^{+\infty}\int_0^T \mathbb{E}\left\{s(t-kT;u_k)s(t-\tau-kT+\ell T;u_{k-\ell})\right\}dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\sum_{\ell=-\infty}^{+\infty}\int_{-kT}^{-kT+T} \mathbb{E}\left\{s(\xi;u_k)s(\xi-\tau+\ell T;u_{k-\ell})\right\}d\xi \\
&= \frac{1}{T}\sum_{\ell=-\infty}^{+\infty}\mathbb{E}\left\{\int_{\mathbb{R}} s(\xi;u_k)s(\xi-\tau+\ell T;u_{k-\ell})\right\}d\xi
\end{aligned}
$$

Power Spectral Density

$$
r_s(\tau;a_m;a_n) \triangleq \int_{\mathbb{R}} s(t;a_m)s(t-\tau;a_n)dt
$$

$$
R_z(\tau) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty}\mathbb{E}\left\{r_s(\tau-\ell T;u_k;u_{k-\ell})\right\}
$$

$$
\begin{aligned}
P_z(f) &= \mathcal{F}\left\{R_z(\tau)\right\}(\tau\rightarrow f) \\
&= \frac{1}{T}\sum_{\ell=\infty}^{+\infty}\mathbb{E}\left\{S(f;u_k)S^*(f;u_{k-\ell})\right\}\exp(-j2\pi f\ell T)
\end{aligned}
$$

PSD for ASK

$$
s(t-kT;u_k) = L_k\, p(t-kT)\,, \qquad L_k \in \{A_1,\dots,A_M\}
$$

$$
r_s(\tau;a_m;a_n) = A_mA_n\int_{\mathbb{R}} p(t)p(t-\tau)dt = A_mA_n r_p(\tau)
$$

$$
R_z(\tau) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty}\mathbb{E}\left\{L_kL_{k-\ell}\right\}r_p(\tau-\ell T) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty}R_L(\ell)r_p(\tau-\ell T)
$$

$$
\begin{aligned}
P_z(f) &= \frac{1}{T}|P(f)|^2\sum_{\ell=-\infty}^{+\infty}R_L(\ell)\exp(-j2\pi f\ell T) \\
&= \frac{1}{T}|P(f)|^2\mathcal{F}\left\{R_L(\ell)\right\}(\ell\rightarrow\nu=fT)
\end{aligned}
$$

PSD for ASK: example 1

Consider L_k i.i.d. sequence with $\mu_L = \mathbb{E}\left\{L_k\right\}$ and $\sigma_L^2 = \mathbb{V}\mathrm{ar}\left\{L_k\right\}$, then

$$
R_L(\ell) = \begin{cases}\sigma_L^2+\mu_L^2 & \ell = 0 \\ \mu_L^2 & \ell \neq 0\end{cases} = \sigma_L^2\delta(\ell)+\mu_L^2
$$

$$
\mathcal{F}\left\{R_L(\ell)\right\}(\ell\rightarrow\nu) = \sigma_L^2+\mu_L^2\sum_{\ell=-\infty}^{+\infty}\exp(-j2\pi\ell\nu) = \sigma_L^2+\mu_L^2\sum_{\ell=-\infty}^{+\infty}\delta(\nu-\ell)
$$

The presence of a non-null mean implies the presence of spectral impulses with periodic spacing

$$
P_z(f) = \frac{\sigma_L^2}{T}|P(f)|^2+\frac{\mu_L^2}{T^2}|P(f)|^2\sum_{\ell=-\infty}^{+\infty}\delta\left(f-\frac{\ell}{T}\right)
$$

Bandwidth

Spectral occupancy depends on the bandwidth of the analog pulse and does not grow with cardinality M

Analog pulse of duration T gives approximate bandwidth of $1/T$

Analog SSB may be used to halve the spectral occupancy through adding quadrature component

PSD and Bandwidth for PSK, ASK/PSK, QAM

PSK, ASK/PSK and QAM may be viewed as complex ASK modulations (using equivalent baseband representation)

The PSD expression is analogous to the ASK case

Quadrature component is not available for pulse shaping, spectral occupancy cannot be halved

PSD for FSK

FSK may be viewed as a sum of M ON-OFF signals

$$z(t) = \sum_{m=1}^{M} z_m(t) \qquad \text{where} \qquad z_m(t-kT) = \begin{cases} z(t-kT) & u_k = a_m \\ 0 & u_k \neq a_m \end{cases}$$

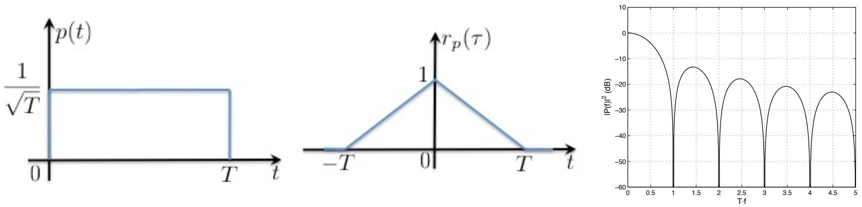
$$\begin{aligned} R_z(\tau) &= \sum_{m=1}^{M}\sum_{n=1}^{M} \mathbb{E}\{z_m(t) z_n(t-\tau)\} \\ &= \sum_{m=1}^{M} R_{z_m}(\tau) + \sum_{m=1}^{M} \sum_{n=1,\ n\neq m} R_{z_m,z_n}(\tau) \\ P_z(f) &\approx \sum_{m=1}^{M} P_{z_m}(f) \\ &= \sum_{m=1}^{M} \frac{|P(f-f_m)|^2 + |P(f+f_m)|^2}{2T} \mathcal{F}\{R_A(\ell)\}\,(\ell \to \nu = fT) \end{aligned}$$

Bandwidth

- FSK is always generated as a CPFSK
- the difference between FSK and CPFSK is in the demodulator
- CPFSK may be viewed as an FM with continuous-time discrete-amplitude modulating signal
- Carson bandwidth may be used as an estimate of the bandwidth

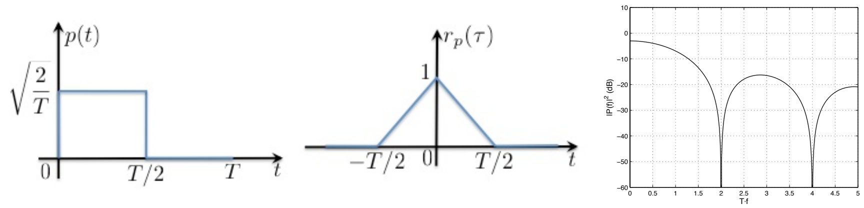
Spectral occupancy depends on the bandwidth of the analog pulse and grows with the cardinality M

Non-Return-to-Zero (NRZ)



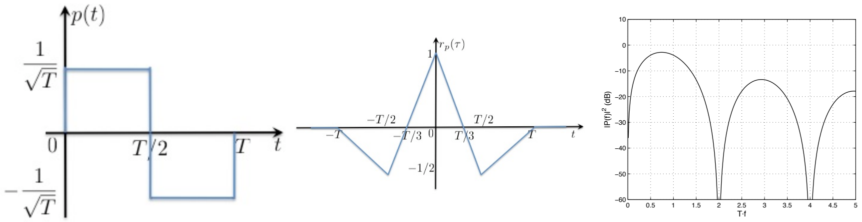
$$
\begin{aligned}
p(t) &= \frac{1}{\sqrt{T}} \mathrm{rect}\left(\frac{t - T/2}{T}\right) \\
r_p(\tau) &= \Lambda\left(\frac{\tau}{T}\right) \\
|P(f)|^2 &= T \mathrm{sinc}^2 (Tf)
\end{aligned}
$$

Return-to-Zero (RZ)



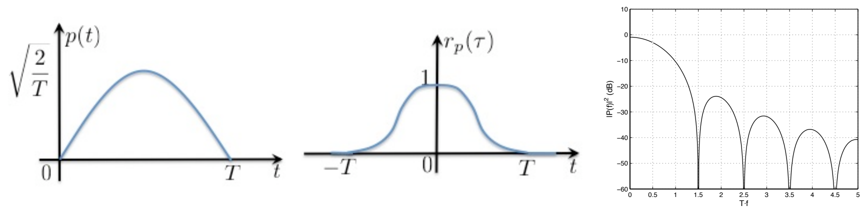
$$
\begin{aligned}
p(t) &= \sqrt{\frac{2}{T}} \mathrm{rect}\left(\frac{t - T/4}{T/2}\right) \\
r_p(\tau) &= \Lambda\left(\frac{\tau}{T/2}\right) \\
|P(f)|^2 &= \frac{T}{2} \mathrm{sinc}^2 \left(\frac{T}{2} f\right)
\end{aligned}
$$

Manchester



$$
\begin{aligned}
p(t) &= \frac{1}{\sqrt{T}} \left(\mathrm{rect}\left(\frac{t - T/4}{T/2}\right) - \mathrm{rect}\left(\frac{t - 3T/4}{T/2}\right) \right) \\
r_p(\tau) &= \frac{3}{2} \Lambda\left(\frac{\tau}{T/2}\right) - \frac{1}{2} + \Lambda\left(\frac{\tau}{T}\right) \left(\mathrm{rect}\left(\frac{\tau}{T}\right) - 1 \right) \\
|P(f)|^2 &= T \sin^2\left(\pi \frac{T}{2} f\right) \mathrm{sinc}^2 \left(\frac{T}{2} f\right)
\end{aligned}
$$

Half-Sine (HS)



$$
\begin{aligned}
p(t) &= \sqrt{\frac{2}{T}} \sin\left(\frac{\pi t}{T}\right) \mathrm{rect}\left(\frac{t - T/2}{T}\right) \\
r_p(\tau) &= \cos\left(\frac{\pi|\tau|}{T}\right) \Lambda\left(\frac{\tau}{T}\right) + \frac{1}{\pi} \sin\left(\frac{\pi|\tau|}{T}\right) \mathrm{rect}\left(\frac{\tau}{2T}\right) \\
|P(f)|^2 &= \frac{T}{2\pi^2} \left(\frac{\cos(\pi T f)}{(Tf)^2 - 1/4} \right)^2
\end{aligned}
$$

PSD Comparison