Digital Communications

— Lecture 09 —
Complex-Valued Modulation

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Outline

System Model

- $\mathcal{A} = \{a_1, \ldots, a_M\}$
- $\Pi = \{p_1, \ldots, p_M\}$
- $\{s_1(t), \ldots, s_M(t)\},\ t \in [0, T)$
- $\mathcal{E}_m$ is the energy of the mth signal

Consider complex-valued signals and noise

$$u = a_m \quad \rightarrow \quad s_m(t) = s_{R,m}(t) + j s_{I,m}(t) \in \mathbb{C} \quad \text{is trasmitted}$$

complex AWGN channel: $w(t) = w_R(t) + j w_I(t) \in \mathbb{C}$
$\mu_{w_R}(t) = \mu_{w_I}(t) = 0$, $P_{w_R}(f) = P_{w_I}(f) = \eta_0/2$, $P_{w_R,w_I}(f) = 0$

The signal space has dimension $N \leq M$ and $\{\psi_1(t), \ldots, \psi_N(t)\}, t \in [0, T)$ is an orthonormal set for signal representation
$\psi_n(t) = \psi_{R,n}(t) + j\psi_{I,n}(t) \in \mathbb{C}$

Signal Constellation

The mth signal $s_m(t)$ is represented by

$$\boldsymbol{s}_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,N} \end{pmatrix} = \begin{pmatrix} < s_m, \psi_1 > \\ \vdots \\ < s_m, \psi_N > \end{pmatrix}$$

where

$$\begin{aligned} s_{m,n} &= \int_0^T s_m(t)\psi_n^*(t)dt = s_{R,m,n} + j s_{I,m,n} \\ &= \int_0^T \left(s_{R,m}(t)\psi_{R,n}(t) - s_{I,m}(t)\psi_{I,n}(t)\right) dt \\ &\quad + j \int_0^T \left(s_{R,m}(t)\psi_{I,n}(t) + s_{I,m}(t)\psi_{R,n}(t)\right) dt \end{aligned}$$

The signal constellation is $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$

Received Signal and Sufficient Statistic

M possible hypotheses in $t \in [0, T)$

$$\begin{aligned} H_m : r(t) &= s_m(t) + w(t) \qquad m = 1, \ldots, M \\ &= \underbrace{(s_{R,m}(t) + w_R(t))}_{r_R(t)} + j \underbrace{(s_{I,m}(t) + w_I(t))}_{r_I(t)} \end{aligned}$$

$$\boldsymbol{r}|H_m = \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,N} \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} = \boldsymbol{s}_m + \boldsymbol{w} \qquad \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{s}_m, \eta_0 \boldsymbol{I}_N)$$

$$f_{\boldsymbol{r}|H_m}(\boldsymbol{r}) = \left(\frac{1}{\pi\eta_0}\right)^N \exp\left(-\frac{1}{\eta_0}\sum_{n=1}^{N}\left((r_{R,n} - s_{R,m,n})^2 + (r_{I,n} - s_{I,m,n})^2\right)\right)$$

Complex-Valued On-Off (1/2)

$M = 2$ possible hypotheses in $t \in [0, T)$: $s_0(t) = 0$ and $s_1(t) = s(t)$

$N = 1$: $\psi(t) = \psi_R(t) + j\psi_I(t)$

$$r = \int_0^T r(t)\psi_n^*(t)dt = \int_0^T (r_R(t) + jr_I(t))(\psi_R(t) - j\psi_I(t))\, dt$$

$$\begin{aligned} r|H_0 &= \int_0^T w(t)\psi_n^*(t)dt \qquad \sim \mathcal{N}_{\mathbb{C}}(0, \eta_0) \\ r|H_1 &= \int_0^T (s(t) + w(t))\psi_n^*(t)dt \qquad \sim \mathcal{N}_{\mathbb{C}}(s, \eta_0) \end{aligned}$$

where

$$s = \int_0^T s(t)\psi_n^*(t)dt = s_r + js_I$$

Complex-Valued On-Off (2/2)

$$\begin{aligned} f_{r|H_0}(r) &= \frac{1}{\pi\eta_0}\exp\left(-\frac{r_R^2}{\eta_0}\right)\exp\left(-\frac{r_I^2}{\eta_0}\right) \\ f_{r|H_1}(r) &= \frac{1}{\pi\eta_0}\exp\left(-\frac{(r_R - s_R)^2}{\eta_0}\right)\exp\left(-\frac{(r_I - s_I)^2}{\eta_0}\right) \end{aligned}$$

MAP decision is

$$\begin{aligned} p_0 f_{r|H_0}(r) &\gtrless p_1 f_{r|H_1}(r) \\ \eta_0\log(p_0) - r_R^2 - r_I^2 &\gtrless \eta_0\log(p_1) - (r_R - s_R)^2 - (r_I - s_I)^2 \\ 2(s_R r_R + s_I r_I) &\gtrless \eta_0 \log\left(\frac{p_0}{p1}\right) + (s_R^2 + s_I^2) \\ \Re(sr) &\gtrless \frac{\eta_0}{2}\log\left(\frac{p_0}{p1}\right) + \frac{\mathcal{E}_s}{2} \end{aligned}$$

MAP decision for M-ary modulation

MAP decision rule is the following

$$\begin{aligned} \hat{u} &= \arg\max_m \{\eta_0\log(p_m) - \|\boldsymbol{r} - \boldsymbol{s}_m\|^2\} \\ &= \arg\max_m \{\eta_0\log(p_m) - \|\boldsymbol{s}_m\|^2 + 2\Re\{\boldsymbol{s}_m^{\mathrm{H}}\boldsymbol{r}\}\} \\ &= \arg\max_m \left\{\Re\{\boldsymbol{s}_m^{\mathrm{H}}\boldsymbol{r}\} - \frac{\mathcal{E}_m}{2} + \frac{\eta_0}{2}\log(p_m)\right\} \end{aligned}$$

The components of the decision vector are

$$\begin{aligned} y_m &= \Re\{\boldsymbol{s}_m^{\mathrm{H}}\boldsymbol{r}\} - \frac{\mathcal{E}_m}{2} + \frac{\eta_0}{2}\log(p_m) \\ &= \Re\left\{\int_0^T \boldsymbol{s}_m^*(t)r(t)dt\right\} - \frac{1}{2}\int_0^T |\boldsymbol{s}_m(t)|^2 dt + \frac{\eta_0}{2}\log(p_m) \end{aligned}$$

Noncoherent Bandpass Modulation

Local oscillators at TX and RX locations are not synchronized

The received signal is

$$z(t, \mathcal{M}, \vartheta) = \sum_{k=-\infty}^{+\infty} s(t - kT; u_k; \vartheta)$$

where

$$s(t - kT; u_k; \vartheta) = \Re\left\{\tilde{s}(t - kT; u_k) e^{j\vartheta} e^{j2\pi f_0 t}\right\}$$

- $\tilde{s}(t; u)$ is the baseband complex signal
- f_0 is the carrier frequency
- $\vartheta \sim \mathcal{U}(0, 2\pi)$ is the phase synchronization error

MAP decision

Assume known ϑ and denote $s_m(t; \vartheta) = \Re\{\tilde{s}_m(t) \exp(j\vartheta) \exp(j2\pi f_0 t)\}$, the conditional MAP decision is based on $p_m \Lambda_m(\theta)$ where

$$\begin{aligned}
\Lambda_m(\vartheta) &= \exp\left(\frac{2}{\eta_0}\int_0^T r(t) s_m(t;\vartheta) dt - \frac{1}{\eta_0}\int_0^T s_m^2(t;\vartheta) dt\right) \\
&= \exp\left(\frac{2}{\eta_0}\int_0^T r(t) s_m(t;\vartheta) dt - \frac{\mathcal{E}_m}{\eta_0}\right) \\
&= \exp\left(\frac{2}{\eta_0}\int_0^T r(t) \Re\left\{\tilde{s}_m(t)\exp(j\vartheta)\exp(j2\pi f_0 t)\right\} dt - \frac{\mathcal{E}_m}{\eta_0}\right) \\
&= \exp\left(\frac{2}{\eta_0}\Re\left\{e^{j\vartheta}\int_0^T r(t)\tilde{s}_m(t) e^{j2\pi f_0 t} dt\right\}\right)\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)
\end{aligned}$$

However ϑ is unknown, thus MAP decision is based on statistical averages

$$\hat{u} = \arg\max_m \{p_m \Lambda_m\} \qquad \text{where} \qquad \Lambda_m = \mathbb{E}_\vartheta\{\Lambda_m(\vartheta)\}$$

Average Likelihood (1/3)

$$\begin{aligned}
\Lambda_m &= \mathbb{E}_\vartheta\{\Lambda_m(\vartheta)\} = \int_{\mathbb{R}} f_\vartheta(\vartheta)\Lambda_m(\vartheta) d\vartheta = \frac{1}{2\pi}\int_0^{2\pi}\Lambda_m(\vartheta) d\vartheta \\
&= \frac{\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)}{2\pi}\int_0^{2\pi}\exp\left(\frac{2}{\eta_0}\Re\left\{e^{j\vartheta}\int_0^T r(t)\tilde{s}_m(t)e^{j2\pi f_0 t} dt\right\}\right) d\vartheta \\
&= \frac{\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)}{2\pi}\int_0^{2\pi}\exp\left(\frac{2}{\eta_0}\Re\left\{e^{j\vartheta}\sqrt{\mathcal{E}_m} L_m^*\right\}\right) d\vartheta
\end{aligned}$$

where we denote

$$L_m = \frac{1}{\sqrt{\mathcal{E}_m}}\int_0^T r(t)\tilde{s}_m^*(t)\exp(-j2\pi f_0 t) dt$$

Average Likelihood (2/3)

From baseband representation we know that

$$\begin{aligned}
\tilde{s}_m(t) &= s_{C,m}(t) - j s_{S,m}(t) \\
s_m(t) &= s_{C,m}(t)\cos(2\pi f_0 t) + s_{S,m}(t)\sin(2\pi f_0 t) \\
s_m^{\angle}(t) &= s_{S,m}(t)\cos(2\pi f_0 t) - s_{C,m}(t)\sin(2\pi f_0 t)
\end{aligned}$$

then

$$\begin{aligned}
L_m &= \frac{1}{\sqrt{\mathcal{E}_m}}\int_0^T r(t)\tilde{s}_m^*(t)\exp(-j2\pi f_0 t) dt \\
&= \frac{1}{\sqrt{\mathcal{E}_m}}\int_0^T r(t)(s_{C,m}(t) + j s_{S,m}(t))(\cos(2\pi f_0 t) - j\sin(2\pi f_0 t)) dt \\
&= \frac{1}{\sqrt{\mathcal{E}_m}}\int_0^T r(t) s_m(t) dt + \frac{j}{\sqrt{\mathcal{E}_m}}\int_0^T r(t) s_m^{\angle}(t) dt
\end{aligned}$$

Average Likelihood (3/3)

$$\begin{aligned}
\Lambda_m &= \frac{\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)}{2\pi}\int_0^{2\pi}\exp\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\cos(\vartheta-\arctan(L_m))\right)d\vartheta \\
&= \frac{\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)}{2\pi}\int_0^{2\pi}\exp\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\cos(\vartheta)\right)d\vartheta \\
&= \frac{\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)}{2\pi} I_0\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\right)
\end{aligned}$$

where the 0-order modified Bessel function of the first kind is

$$I_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\exp(\pm z\cos(\vartheta))d\vartheta$$

MAP and ML decision

MAP decision

$$\begin{aligned}
\hat{u} &= \arg\max_m\left\{p_m\exp\left(-\frac{\mathcal{E}_m}{\eta_0}\right)I_0\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\right)\right\} \\
&= \arg\max_m\left\{\log(p_m)-\frac{\mathcal{E}_m}{\eta_0}+\log\left(I_0\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\right)\right)\right\}
\end{aligned}$$

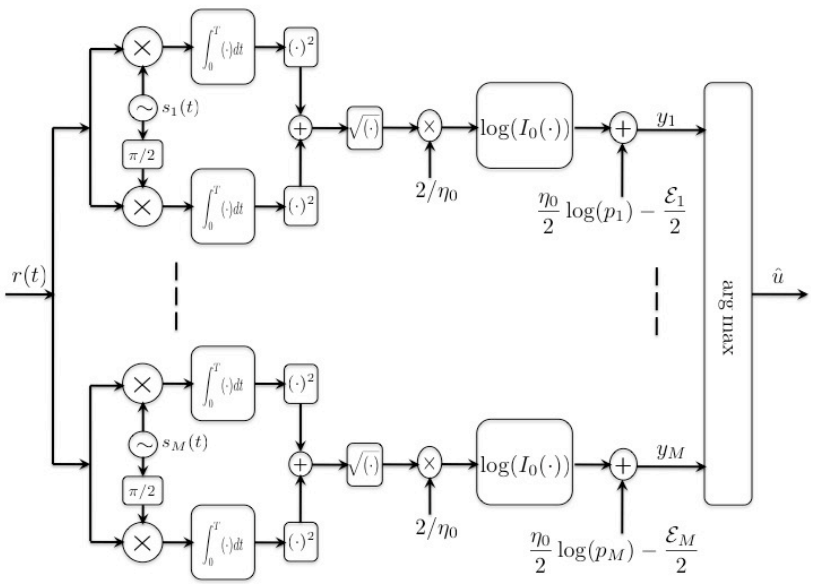
ML decision

$$\hat{u} = \arg\max_m\left\{-\frac{\mathcal{E}_m}{\eta_0}+\log\left(I_0\left(\frac{2\sqrt{\mathcal{E}_m}}{\eta_0}|L_m|\right)\right)\right\}$$

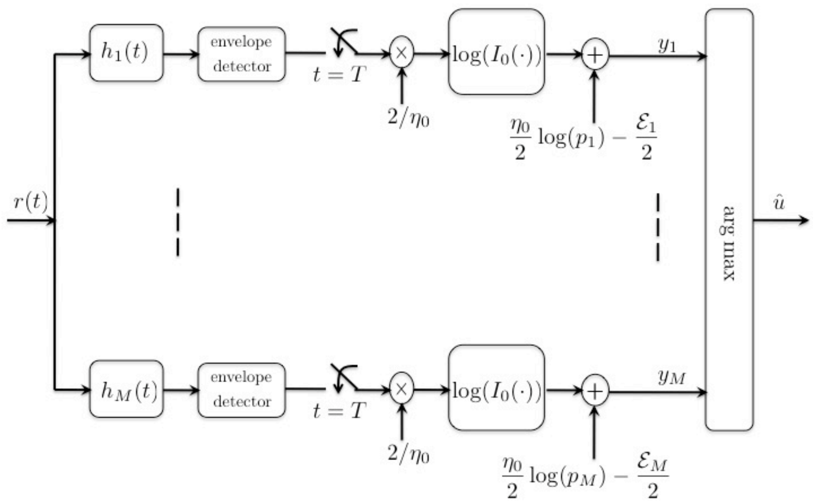
ML decision with equal-energy signals

$$\hat{u} = \arg\max_m\left\{|L_m|^2\right\}$$

MAP Receiver Architecture (1/2)



MAP Receiver Architecture (2/2)



Correlator and Matched Filter Equivalence

Assume $h_m(t) = \Re\{\tilde{s}^*(T-t)\exp(j2\pi f_0 t)\}$ then

$$
\begin{aligned}
y_m(t) &= r(t) \star h(t) = \int_0^T r(t-\tau) h_m(\tau) d\tau \\
&= \int_0^T r(t-\tau) \Re\{\tilde{s}^*(T-\tau)\exp(j2\pi f_0 \tau)\} d\tau \\
&= \Re\left\{\exp(j2\pi f_0 T) \int_0^T r(t-T+\tau)\tilde{s}^*(\tau)\exp(-j2\pi f_0 \tau) d\tau\right\} \\
y_m(t=T) &= \Re\left\{\exp(j2\pi f_0 T)\sqrt{\mathcal{E}_m} L_m\right\}
\end{aligned}
$$

Using the envelope detector provides $\sqrt{\mathcal{E}_m}|L_m|$

OOK (1/2)

$M = 2$

$$
\begin{aligned}
s_0(t) &= 0 & \tilde{s}_0(t) &= 0 \\
s_1(t) &= \sqrt{\frac{2\mathcal{E}}{T}}\cos(2\pi f_0 t) & \tilde{s}_1(t) &= \sqrt{\frac{2\mathcal{E}}{T}}\mathrm{rect}\left(\frac{t-T/2}{T}\right)
\end{aligned}
$$

$$
L_0 = 0 \qquad L_1 = \frac{2}{T}\int_0^T r(t)\exp(-j2\pi f_0 t) dt
$$

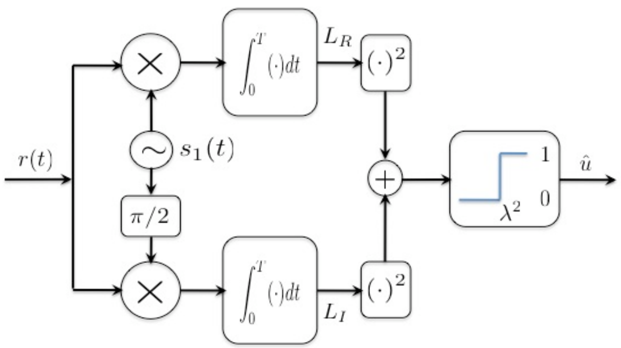
MAP decision is

$$
\begin{aligned}
\log(p_0) &\gtrless \log(p_1) - \frac{\mathcal{E}}{\eta_0} + \log\left(I_0\left(\frac{2\sqrt{\mathcal{E}}}{\eta_0}|L_1|\right)\right) \\
|L_1| &\gtrless \frac{\eta_0}{2\sqrt{\mathcal{E}}} I_0^{-1}\left(\frac{p_0}{p_1}\exp\left(\frac{\mathcal{E}}{\eta_0}\right)\right)
\end{aligned}
$$

OOK (2/2)

Denoting $\lambda = \frac{\eta_0}{2} I_0^{-1}\left(\frac{p_0}{p_1}\exp\left(\frac{\mathcal{E}}{\eta_0}\right)\right)$, MAP decision is

$$
\mathcal{E}|L_1|^2 \gtrless \lambda^2
$$



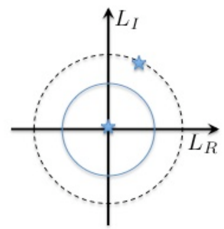
BER for OOK (1/2)

Statistics of decision variables are

$$
\begin{aligned}
(L_R, L_I)^{\mathrm{T}}|H_0 &\sim \mathcal{N}\left(\mathbf{0}, \frac{\eta_0}{2}\boldsymbol{I}\right) \\
(L_R, L_I)^{\mathrm{T}}|H_0 &\sim \mathcal{N}\left(\sqrt{\mathcal{E}}\left(\cos(\vartheta), \sin(\vartheta)\right)^{\mathrm{T}}, \frac{\eta_0}{2}\boldsymbol{I}\right)
\end{aligned}
$$

$P_e = p_0 \Pr(e|H_0) + p_1 \Pr(e|H_1)$

Assume an arbitrary threshold ν^2



$$
\begin{aligned}
\Pr(e|H_0) &= \int_{\mathbb{R}} d\vartheta f_\vartheta(\vartheta) \iint_{\Omega_1} d\alpha_R d\alpha_I f_{L_R,L_I|H_0}(\alpha_R, \alpha_I) \\
\Pr(e|H_1) &= \int_{\mathbb{R}} d\vartheta f_\vartheta(\vartheta) \iint_{\Omega_0} d\alpha_R d\alpha_I f_{L_R,L_I|H_1}(\alpha_R, \alpha_I)
\end{aligned}
$$

BER for OOK (2/2)

$$\begin{aligned}
\Pr(e|H_0) &= \int_0^{2\pi} d\vartheta \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_\nu^\infty d\rho \frac{\rho}{\pi\eta_0} \mathrm{e}^{-\rho^2/\eta_0} \\
&= \exp\left(-\frac{\nu^2}{\eta_0}\right) \\
\Pr(e|H_1) &= \int_0^{2\pi} \frac{d\vartheta}{2\pi} \int_0^{2\pi} d\varphi \int_0^\nu \frac{d\rho\rho}{\pi\eta_0} \mathrm{e}^{-\frac{(\rho\cos(\varphi)-\sqrt{\mathcal{E}}\cos(\vartheta))^2+(\rho\sin(\varphi)-\sqrt{\mathcal{E}}\sin(\vartheta))^2}{\eta_0}} \\
&= \int_0^\nu d\rho \frac{2\rho}{\eta_0} \exp\left(-\frac{\rho^2+\mathcal{E}}{\eta_0}\right) I_0\left(\frac{2\sqrt{\mathcal{E}}\rho}{\eta_0}\right) \\
&= 1 - Q\left(\sqrt{\frac{2\mathcal{E}}{\eta_0}}; \sqrt{\frac{2\nu^2}{\eta_0}}\right)
\end{aligned}$$

Finally

$$P_e = p_0 \exp\left(-\nu^2/\eta_0\right) + (1-p_0)(1 - Q\left(\sqrt{2\mathcal{E}/\eta_0}; \sqrt{2\nu^2/\eta_0}\right))$$