Digital Communications

— Lecture 12 —
Performance over Fading Channels
and Diversity Techniques

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Outline

Flat-Flat Fading

The signal model is

$$r(t) = Hs(t) + w(t)$$

where $H = \alpha \exp(j\vartheta)$ is a complex-valued r.v.

Rayleigh fading is a common and simple model:

- $H \sim \mathcal{N}_{\mathbb{C}}(0, 2\sigma_h^2)$, i.e. $\Re\{H\}, \Im\{H\} \sim \mathcal{N}(0, \sigma_h^2)$
- $\alpha \sim \mathcal{Rayleigh}(\sigma_h)$
- $\vartheta \sim \mathcal{U}(-\pi, \pi)$
- usually $\mathbb{E}\{|H|^2\} = \mathbb{E}\{\alpha^2\} = 2\sigma_h^2 = 1$

The effect of fading is mainly changing the receive SNR into a r.v.

$$\gamma_b = \alpha^2 \mathcal{E}_b / \eta_0$$

Intuition

Consider a BPSK over a channel with average SNR $\overline{\gamma_b} = 10$ dB such that

- half of the time is in "state 0" with $\gamma_b = -\infty$ dB
- half of the time is in "state 1" with $\gamma_b = 13$ dB

The BERs corresponding to the two channel states are

- $\Pr(e|\text{state } 0) = 1/2$
- $\Pr(e|\text{state } 1) = 10^{-10}$

The average BER is

$$P_e = \frac{1}{2}\Pr(e|\text{state } 0) + \frac{1}{2}\Pr(e|\text{state } 1) \simeq 0.25$$

The BER for BPSK over an AWGN channel with SNR $\gamma_b = \overline{\gamma_b} = 10$ dB is

$$P_e \simeq 4 \cdot 10^{-6}$$

Noncoherent Detection (1/2)

$$\mathcal{H}_m: \quad \boldsymbol{r} = \underbrace{\alpha e^{j\theta}}_{H} \boldsymbol{s}_m + \boldsymbol{w}$$

$$\begin{aligned}
f_{\boldsymbol{r}|\mathcal{H}_m,\alpha,\theta}(\boldsymbol{r}) &= \left(\frac{1}{\pi\eta_o}\right)^N \exp\left(-\frac{\|\boldsymbol{r}-\alpha e^{j\theta}\boldsymbol{s}_m\|^2}{\eta_o}\right) \\
&= \frac{\exp\left(\frac{\|\boldsymbol{r}\|^2+\alpha^2\mathcal{E}_m}{\eta_o}\right)}{(\pi\eta_o)^N} \exp\left(-\frac{2}{\eta_o}\Re\{\alpha e^{-j\theta}\boldsymbol{s}_m^{\mathrm{H}}\boldsymbol{r}\}\right) \\
&= \frac{\exp\left(\frac{\|\boldsymbol{r}\|^2+\alpha^2\mathcal{E}_m}{\eta_o}\right)}{(\pi\eta_o)^N} \exp\left(-\frac{2\alpha\|\boldsymbol{r}\|\sqrt{\mathcal{E}_m}}{\eta_o}\cos(\angle\boldsymbol{r}-\angle\boldsymbol{s}_m-\theta)\right)
\end{aligned}$$

$$f_{\boldsymbol{r}|\mathcal{H}_m,\alpha}(\boldsymbol{r}) = \frac{1}{2\pi}\int_0^{2\pi} f_{\boldsymbol{r}|\mathcal{H}_m,\alpha,\theta}(\boldsymbol{r})d\theta = \frac{\exp\left(\frac{\|\boldsymbol{r}\|^2+\alpha^2\mathcal{E}_m}{\eta_o}\right)}{(\pi\eta_o)^N} I_0\left(\frac{2\alpha\|\boldsymbol{r}\|\sqrt{\mathcal{E}_m}}{\eta_o}\right)$$

Noncoherent Detection (2/2)

$$\begin{aligned}
f_{\boldsymbol{r}|\mathcal{H}_m}(\boldsymbol{r}) &= \int_0^\infty f_\alpha(a) f_{\boldsymbol{r}|\mathcal{H}_m,a}(\boldsymbol{r})da \\
&= \int_0^\infty f_\alpha(a) \frac{\exp\left(\frac{\|\boldsymbol{r}\|^2+a^2\mathcal{E}_m}{\eta_o}\right)}{(\pi\eta_o)^N} I_0\left(\frac{2a\|\boldsymbol{r}\|\sqrt{\mathcal{E}_m}}{\eta_o}\right) da
\end{aligned}$$

The ML decision is

$$\hat{u} = \arg\max_m \{f_{\boldsymbol{r}|\mathcal{H}_m}(\boldsymbol{r})\}$$

There is no dependence on the phase of the transmitted signal ($\angle\boldsymbol{s}_m$)
Constant-modulus modulations (e.g. PSK and also QAM) cannot be considered unless performing

- channel estimation
- coherent detection

Coherent Detection (1/2)

- We assume that the specific realization of the r.v. $H = \alpha\exp(j\theta)$ is known at the receiver
- This assumption is commonly denoted Channel State Information at the Receiver (CSIR)

$$\begin{aligned}
\hat{u} &= \arg\max_m \{f_{\boldsymbol{r}|\mathcal{H}_m,\alpha,\theta}(\boldsymbol{r})\} \\
&= \arg\min_m \left\{\|\boldsymbol{r}-\alpha e^{j\theta}\boldsymbol{s}_m\|^2\right\} \\
&= \arg\min_m \left\{\alpha^2\mathcal{E}_m - 2\alpha\sqrt{\mathcal{E}_m}\|\boldsymbol{r}\|\cos(\angle\boldsymbol{r}-\angle\boldsymbol{s}_m-\theta)\right\}
\end{aligned}$$

Binary modulation with equal-energy signals

$$\begin{aligned}
\hat{u} &= \arg\max_{m\in\{1,2\}} \left\{\Re\{\alpha e^{-j\theta}\boldsymbol{s}_m^{\mathrm{H}}\boldsymbol{r}\}\right\} \\
&= \arg\max_{m\in\{1,2\}} \{\cos(\angle\boldsymbol{r}-\angle\boldsymbol{s}_m-\theta)\}
\end{aligned}$$

Coherent Detection (2/2)

Compute the average BER considering $\mathcal{H}_1$, as $P_e = P_{e|\mathcal{H}_1}$
The error event is

$$\begin{aligned}
\|\boldsymbol{r}-\alpha e^{j\theta}\boldsymbol{s}_1\|^2 &> \|\boldsymbol{r}-\alpha e^{j\theta}\boldsymbol{s}_2\|^2 \\
\|\alpha e^{j\theta}\boldsymbol{s}_1+\boldsymbol{w}-\alpha e^{j\theta}\boldsymbol{s}_1\|^2 &> \|\alpha e^{j\theta}\boldsymbol{s}_1+\boldsymbol{w}-\alpha e^{j\theta}\boldsymbol{s}_2\|^2 \\
\|\boldsymbol{w}\|^2 &> \|\alpha e^{j\theta}(\boldsymbol{s}_1-\boldsymbol{s}_2)+\boldsymbol{w}\|^2 \\
\Re\{e^{-j\theta}(\boldsymbol{s}_1-\boldsymbol{s}_2)^{\mathrm{H}}\boldsymbol{w}\} &< -\frac{\alpha}{2}\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2
\end{aligned}$$

Remember that

$$\begin{aligned}
\Re\{e^{-j\theta}(\boldsymbol{s}_1-\boldsymbol{s}_2)^{\mathrm{H}}\boldsymbol{w}\} &\sim \mathcal{N}\left(0,\eta_o\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2\right) \\
\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2 &= 2(1-\rho)2\mathcal{E}
\end{aligned}$$

then if we define $\gamma_b = \alpha^2\mathcal{E}/\eta_o$

$$P_{e|\alpha} = \mathcal{Q}\left(\sqrt{(1-\rho)\frac{\alpha^2\mathcal{E}}{\eta_o}}\right) = \mathcal{Q}\left(\sqrt{(1-\rho)\gamma_b}\right)$$

Average BER

- The SNR is now a r.v. thus the BER is a r.v.
- The average BER is an important performance measure

Average BER may be computed as follows:

- Compute the BER w.r.t. to a given SNR as done for AWGN ($P_e(\gamma_b)$)
- Compute the pdf of the SNR ($f_{\gamma_b}(\gamma_b)$)
- Use the total probability theorem

$$P_e = \int_{\mathbb{R}} f_{\gamma_b}(g) P_e(g) dg$$

SNR distribution

- Rayleigh fading:
  SNR is exponentially distributed $\gamma_b \sim \mathcal{E}xp(1/\Gamma_b)$

$$f_{\gamma_b}(\gamma_b) = \frac{1}{\Gamma_b} \exp\left(-\frac{\gamma_b}{\Gamma_b}\right) u(\gamma_b) \qquad \Gamma_b = (2\sigma_h^2)\frac{\mathcal{E}_b}{\eta_o}$$

- Ricean fading:

$$f_{\gamma_b}(\gamma_b) = \frac{1+K}{\Gamma_b} \exp\left(-\frac{(1+K)\gamma_b}{\Gamma_b} - K\right) I_0\left(\sqrt{\frac{4(1+K)K\gamma_b}{\Gamma_b}}\right) u(\gamma_b)$$

- Nakagami fading:

$$f_{\gamma_b}(\gamma_b) = \frac{1}{\Gamma_b} \exp\left(-\frac{\gamma_b}{\Gamma_b}\right) u(\gamma_b)$$

Binary Modulations over Rayleigh Fading (1/3)

- BPSK
- BER for a given SNR is $P_e(\gamma_b) = \mathcal{Q}(\sqrt{2\gamma_b})$
- Average BER is

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\Gamma_b}{1+\Gamma_b}}\right) \approx \frac{1}{4\Gamma_b}$$

- BFSK
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- Average BER is

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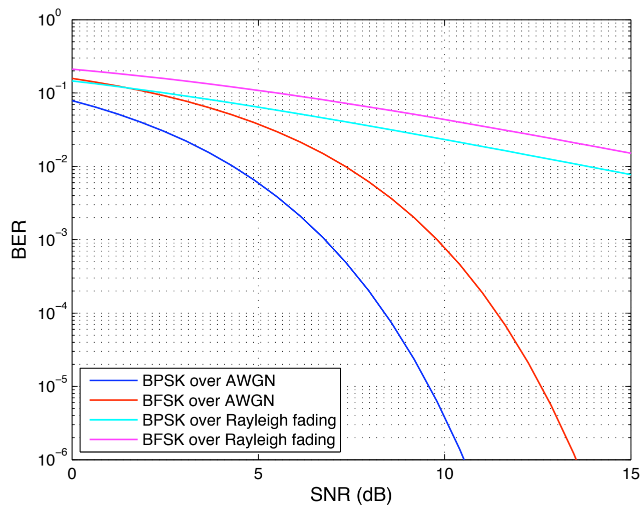
Binary Modulations over Rayleigh Fading (2/3)



Representations for the $\mathcal{Q}$-function

- The classical definition of the $\mathcal{Q}$-function is

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{t^2}{2}\right) dt$$

and corresponds to $\Pr\left(\mathcal{N}(0,1) > x\right)$

- The problem is that the argument x is in the integration limit and not in the integrand function
- An alternative representation is

$$\mathcal{Q}(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2(\theta)}\right) d\theta$$

Moment Generating Function

- Consider a r.v. X distributed according to $f_X(x)$ with $f_X(x) = 0 \quad \forall x < 0$
- The Moment Generating Function (MGF) of X is defined as the Laplace transform of its pdf with reversed argument sign

$$\begin{aligned} \Phi_X(s) &= \mathcal{L}\{f_X(x)\}(-s) = \mathbb{E}\{\exp(sX)\} \\ &= \int_0^{\infty} f_X(x) \exp(sx)\, dx \end{aligned}$$

- The nth moment of X is obtained as

$$\mathbb{E}\{X^n\} = \int_0^{\infty} x^n f_X(x) dx = \left.\frac{\partial^n}{\partial s^n} \Phi_X(s)\right|_{s=0}$$

Moment Generating Function of the SNR

- Rayleigh fading

$$\Phi_{\gamma_b}(s) = \frac{1}{1 - \Gamma_b s}$$

- Ricean fading

$$\Phi_{\gamma_b}(s) = \frac{1+K}{1+K-\Gamma_b s} \exp\left(\frac{K\Gamma_b s}{1+K-\Gamma_b s}\right)$$

- Nakagami fading

$$\Phi_{\gamma_b}(s) = \left(1 - \frac{\Gamma_b}{m} s\right)^{-m}$$

Alternative Method for Computing the Average BER

- Assume the following BER for a given SNR

$$P_e(\gamma_b) = \alpha \exp(-\beta \gamma_b)$$

- then it is straightforward to get

$$P_e = \alpha \Phi_{\gamma_b}(-\beta)$$

- Assume the following BER for a given SNR

$$P_e(\gamma_b) = \int_{c_1}^{c_2} \alpha \exp(-\beta(x)\gamma_b) dx$$

- then it is straightforward to get

$$P_e = \alpha \int_{c_1}^{c_2} \Phi_{\gamma_b}(-\beta(x)) dx$$

Binary Modulations over Rayleigh Fading (3/3)

- BPSK
- $\alpha = 1/\pi$, $\beta(\theta) = 1/\sin^2(\theta)$, $c_1 = 0$, and $c_2 = \pi/2$
- Average BER is

$$P_e = \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{1+\frac{\Gamma_b}{\sin^2(\theta)}} d\theta$$

- BFSK
- $\alpha = 1/\pi$, $\beta(\theta) = 1/2\sin^2(\theta)$, $c_1 = 0$, and $c_2 = \pi/2$
- Average BER is

$$P_e = \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{1+\frac{\Gamma_b}{2\sin^2(\theta)}} d\theta$$

Outage Probability

- Average BER is meaningful when $T \approx T_c$,
  i.e. a block of transmitted information undergoes many different channel realizations which fades almost independently
- Outage Probability (OP) is meaningful when $T \ll T_c$,
  i.e. a block of transmitted information undergoes the same channel realization and fading causes error bursts

OP denotes the probability that the SNR falls below a given threshold

$$P_{out} = \Pr(\gamma_b < \gamma_0) = \int_{-\infty}^{\gamma_0} f_{\gamma_b}(g)dg$$

For Rayleigh fading the outage probability is

$$P_{out} = 1 - \exp\left(-\frac{\gamma_0}{\Gamma_b}\right)$$

Channel Estimation (1/2)

- In order to employ coherent detection we need CSIR
- One possibility is to transmit pilot symbols (known at the receiver)
- Pilot symbols must be orthogonal in time or frequency to the data
- Pilot symbols must be less than coherence time or coherence bandwidth away from the data

Denote $u_p(t)$ and $v_p(t)$ the pth pilot symbol and the corresponding received signal

$$v_p(t) = H u_p(t) + w_p(t)\ , \qquad p = 1, \ldots, P$$

A possible estimator could be

$$\hat{H} = \frac{\sum_{p=1}^{P}\int_0^T v_p(t)u_p^*(t)dt}{\sum_{p=1}^{P}\int_0^T |u_p(t)|^2 dt} = H + \frac{\sum_{p=1}^{P}\int_0^T w_p(t)u_p^*(t)dt}{\sum_{p=1}^{P}\int_0^T |u_p(t)|^2 dt}$$

Channel Estimation (2/2)

The estimation error is defined as

$$\varepsilon = \hat{H} - H = \frac{\sum_{p=1}^{P}\int_0^T w_p(t)u_p^*(t)dt}{\sum_{p=1}^{P}\int_0^T |u_p(t)|^2 dt}$$

It has the following interesting properties

$$\begin{aligned}
\mathbb{E}\{\varepsilon\} &= 0 \\
\varepsilon_{rms}^2 &= \mathbb{E}\{\varepsilon^2\} \\
&= \left(\frac{1}{\sum_{p=1}^{P}\mathcal{E}_p}\right)^2 \sum_{p=1}^{P}\sum_{q=1}^{P}\int_0^T dt \int_0^T d\tau\ u_p^*(t)u_q(\tau)\mathbb{E}\{w_p(t)w_q^*(\tau)\} \\
&= \frac{\eta_o}{\sum_{p=1}^{P}\mathcal{E}_p} = \frac{1}{\mathrm{SNR}_p}
\end{aligned}$$

Diversity

Provide the receiver with several replicas of the same information each undergoing an (ideally) independent channel realization

- denote p the probability that the signal fades below a given threshold
- the probability that L independent replicas of the same information all fade below the same threshold is p^L

Most popular forms of diversity are:

- Time diversity
- Frequency diversity
- Space diversity
  - Transmit diversity
  - Receive diversity

Coherent Detection with Diversity (1/3)

The signal model for L independent diversity branches is

$$\mathcal{H}_m : \qquad \boldsymbol{r}_\ell = \underbrace{\alpha_\ell e^{j\theta_\ell}}_{H_\ell} \boldsymbol{s}_\ell(m) + \boldsymbol{w}_\ell \,, \qquad \ell = 1, \ldots, L$$

ML decision is

$$\begin{aligned}\hat{u} &= \arg\max_m \left\{ f_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_L|\mathcal{H}_m,\alpha_1,\ldots,\alpha_L,\theta_1,\ldots,\theta_L}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_L) \right\} \\ &= \arg\min_m \left\{ \sum_{\ell=1}^{L} \|\boldsymbol{r}_\ell - \alpha_\ell e^{j\theta_\ell} \boldsymbol{s}_\ell(m)\|^2 \right\} \\ &= \arg\min_m \left\{ \sum_{\ell=1}^{L} \left(\alpha_\ell^2 \mathcal{E}_\ell(m) - 2\Re\{\alpha_\ell e^{-j\theta_\ell} \boldsymbol{s}_\ell^{\mathrm{H}}(m) \boldsymbol{r}_\ell\} \right) \right\}\end{aligned}$$

Binary modulation with equal-energy signals

$$\hat{u} = \arg\max_{m\in\{1,2\}} \left\{ \sum_{\ell=1}^{L} \Re\{\alpha_\ell e^{-j\theta_\ell} \boldsymbol{s}_\ell^{\mathrm{H}}(m) \boldsymbol{r}_\ell\} \right\}$$

Coherent Detection with Diversity (2/3)

Compute the average BER considering $\mathcal{H}_1$, i.e. $\boldsymbol{r}_\ell = \alpha_\ell e^{j\theta_\ell} \boldsymbol{s}_\ell(1) + \boldsymbol{w}_\ell$, as $P_e = P_{e|\mathcal{H}_1}$

The error event is

$$\begin{aligned}\sum_{\ell=1}^{L} \|\boldsymbol{r}_\ell - \alpha_\ell e^{j\theta_\ell} \boldsymbol{s}_\ell(1)\|^2 &> \sum_{\ell=1}^{L} \|\boldsymbol{r}_\ell - \alpha_\ell e^{j\theta_\ell} \boldsymbol{s}_\ell(2)\|^2 \\ \sum_{\ell=1}^{L} \|\boldsymbol{w}_\ell\|^2 &> \sum_{\ell=1}^{L} \|\alpha_\ell e^{j\theta_\ell} (\boldsymbol{s}_\ell(1) - \boldsymbol{s}_\ell(2)) + \boldsymbol{w}_\ell\|^2 \\ \sum_{\ell=1}^{L} \Re\{\alpha_\ell e^{-j\theta_\ell} (\boldsymbol{s}_\ell(1) - \boldsymbol{s}_\ell(2))^{\mathrm{H}} \boldsymbol{w}_\ell\} &< -\frac{1}{2} \sum_{\ell=1}^{L} \alpha_\ell^2 \|\boldsymbol{s}_\ell(1) - \boldsymbol{s}_\ell(2)\|^2\end{aligned}$$

Remember that

$$\begin{aligned}\Re\{\alpha_\ell e^{-j\theta} (\boldsymbol{s}_1 - \boldsymbol{s}_2)^{\mathrm{H}} \boldsymbol{w}\} &\sim \mathcal{N}\left(0, \alpha_\ell^2 \eta_o \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2\right) \\ \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2 &= 2(1-\rho)2\mathcal{E}\end{aligned}$$

Coherent Detection with Diversity (3/3)

Then

$$\sum_{\ell=1}^{L} \Re\{\alpha_\ell e^{-j\theta} (\boldsymbol{s}_1 - \boldsymbol{s}_2)^{\mathrm{H}} \boldsymbol{w}\} \sim \mathcal{N}\left(0, 2(1-\rho)2\mathcal{E}\eta_o \sum_{\ell=1}^{L} \alpha_\ell^2\right)$$

If we define

$$\begin{aligned}\alpha_L^2 &= \sum_{\ell=1}^{L} \alpha_\ell^2 \\ \gamma_b &= \alpha_L^2 \mathcal{E} / \eta_o\end{aligned}$$

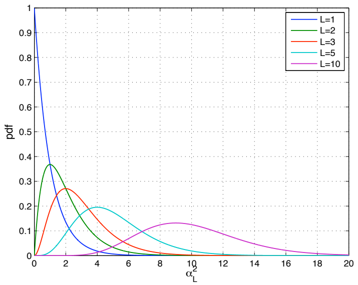
then

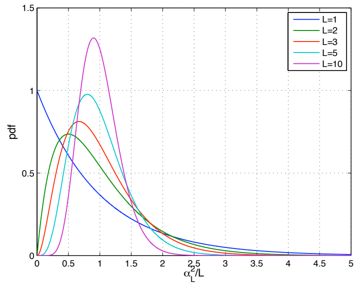
$$P_{e|\alpha_L} = \mathcal{Q}\left(\sqrt{(1-\rho)\frac{\alpha_L^2 \mathcal{E}}{\eta_o}} \right) = \mathcal{Q}\left(\sqrt{(1-\rho)\gamma_b} \right)$$

Erlang Distribution

For Rayleigh i.i.d. channels $\alpha_\ell \sim Rayleigh(1/\sqrt{2})$, i.e. $\alpha_\ell^2 \sim Exp(1)$
It is easy to show that $\alpha_L^2 = \sum_{\ell=1}^L \alpha_\ell^2 \sim Erlang(L,1)$, i.e.

$$\begin{aligned} f_{\alpha_L^2}(\xi) &= \frac{1}{(L-1)!}\xi^{L-1}\exp(-\xi)u(\xi) \\ f_{\alpha_L^2/L}(\xi) &= \frac{L^L}{(L-1)!}\xi^{L-1}\exp(-L\xi)u(\xi) \end{aligned}$$





Binary Modulations over Rayleigh Fading with Diversity

Define

$$\begin{aligned} \gamma_b &= \left(\frac{\mathcal{E}}{\eta_o}\right)\frac{\alpha_L^2}{L} = \Gamma_b\frac{\alpha_L^2}{L} \\ \Gamma_b &= \mathbb{E}\{\gamma_b\} = \frac{\mathcal{E}}{\eta_o} \end{aligned}$$

then

$$f_{\gamma_b}(\gamma) = \left(\frac{L}{\Gamma_b}\right)^L \frac{\gamma^{L-1}}{(L-1)!}\exp\left(-\frac{L}{\Gamma_b}\gamma\right)u(\gamma)$$

and

$$P_e = \int_0^\infty f_{\gamma_b}(\gamma)\mathcal{Q}\left(\sqrt{(1-\rho)L\gamma}\right)d\gamma$$

BPSK over Rayleigh Fading with Diversity