PALMIERI - I0

NOTE INTRODUTTIVE ALLA ELABORAZIONE DELLE IMMAGINI

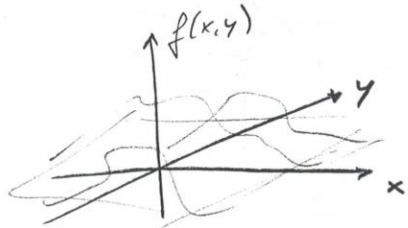
lezioni del corso di
TRASMISSIONE ED ELABORAZIONE NUMERICA
DEI SEGNALI / COMUNICAZIONI ELETTRICHE

A.A 2016-17

(SUN)

Prof. FRANCESCO A.N. PALMIERI

TWO-DIMENSIONAL SIGNALS



$f(x,y)$ - (B/W)

color: $\begin{cases} f_R(x,y) \\ f_G(x,y) \\ f_B(x,y) \end{cases}$

We restrict our attention to B/W pictures (intensity).

THE TWO-DIMENSIONAL FOURIER TRANSFORM:

$$\boxed{F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy}$$

(u,v) [Hertz] 2D-Frequency domain.

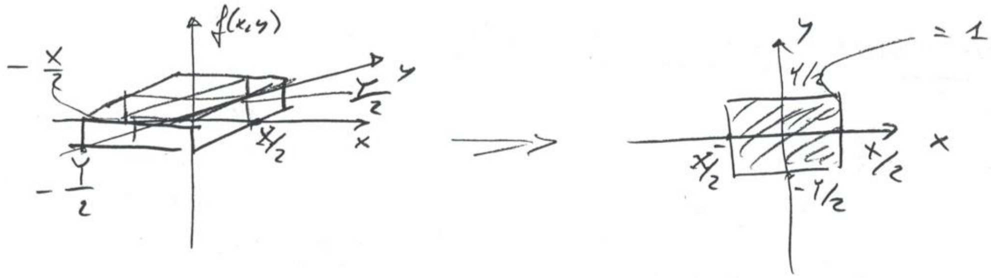
INVERSE:

$$\boxed{f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv}$$

Most of the properties derived for the one-dimensional Fourier Transform can be easily translated in their 2D-version.

Let us consider first some typical 2-D functions and their transforms:

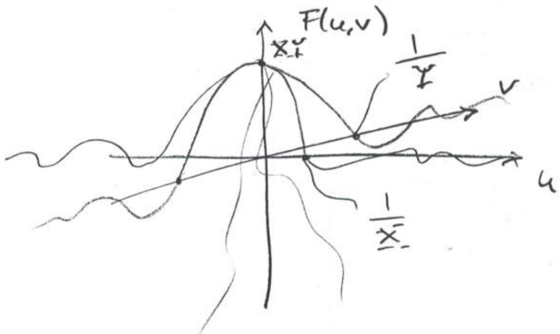
THE 2D-RECTANGULAR FUNCTION



$$rect\left[\frac{x}{X}, \frac{y}{Y}\right] = \Pi\left(\frac{x}{X}\right)\Pi\left(\frac{y}{Y}\right)$$

$$F(u,v) = \iint \Pi\left(\frac{x}{X}\right)\Pi\left(\frac{y}{Y}\right)e^{-j2\pi ux}\,e^{-j2\pi vy}\,dx\,dy$$

$$= XY \operatorname{sinc} Xu \operatorname{sinc} Yv$$

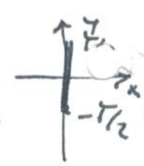


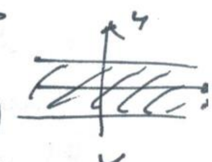
Two-dimensional sinc function.

Consider degenerate

$f(x,y) = \delta(x)\,\Pi\left(\frac{y}{Y}\right)$

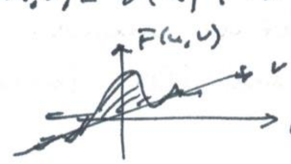
$F(u,v) = Y \operatorname{sinc} Yv$





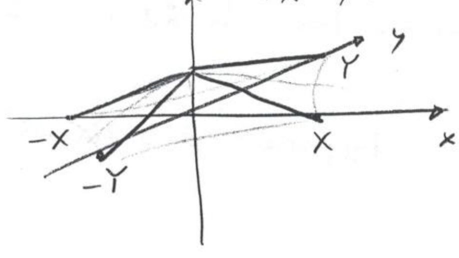
$f(x,y) = \Pi\left(\frac{y}{Y}\right)$

$F(u,v) = \delta(u)\,Y \operatorname{sinc} Yv$



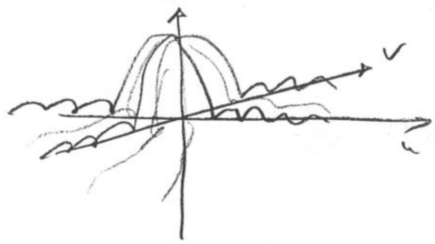
THE 2-D TRIANGULAR FUNCTION

$$\Lambda\left(\frac{x}{X}, \frac{y}{Y}\right)$$

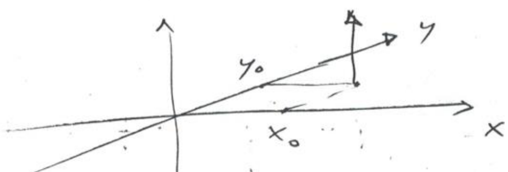


$$\Lambda\left(\frac{x}{X},\frac{y}{Y}\right) = \Lambda\left(\frac{x}{X}\right)\Lambda\left(\frac{y}{Y}\right)$$

$$F(u,v) = XY \operatorname{sinc}^2 Xu \operatorname{sinc}^2 Yv$$



THE 2-D DIRAC FUNCTION



$$\delta(x-x_0,\, y-y_0)$$

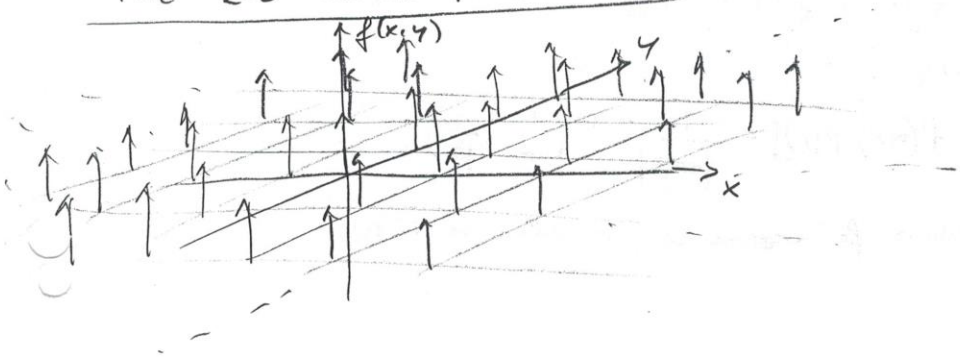
PROPERTIES

$$\left\{ \begin{array}{l} \iint \delta(x-x_0,\, y-y_0)\, dx\, dy = 1 \\ \iint \delta(x-x_0,\, y-y_0)\, f(x,y)\, dx\, dy \\ \quad = f(x_0, y_0) \end{array} \right.$$

$$F(u,v) = \iint \delta(x-x_0,\, y-y_0)\, e^{-j2\pi(xu+yv)}\, dx\, dy$$

$$= e^{-j2\pi(x_0 u + y_0 v)}$$

THE 2-D COMB FUNCTION



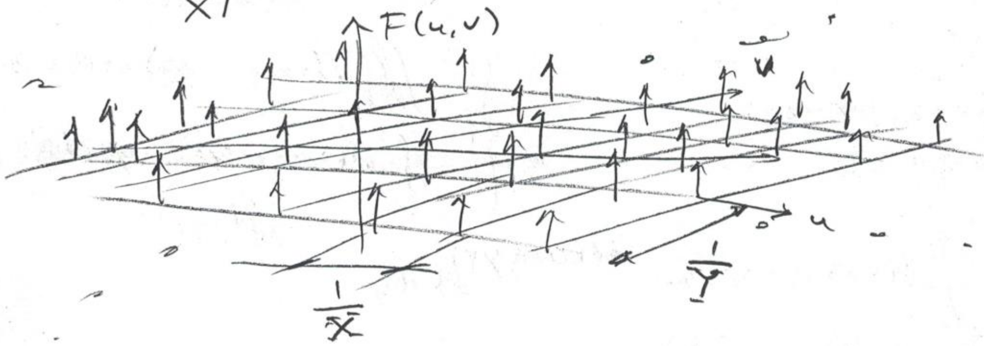
$$\text{comb}_{XY}(x,y) = \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} \delta(x-nX, y-mY)$$

$$\mathcal{F}\left[\text{comb}_{XY}(x,y)\right] = \mathcal{F}\left[\sum_{n=-\infty}^{+\infty}\delta(x-nX)\sum_{m=-\infty}^{+\infty}\delta(y-mY)\right]$$

$$= \frac{1}{X}\sum_{\ell=-\infty}^{+\infty}\delta\left(u-\frac{\ell}{X}\right)\frac{1}{Y}\sum_{k=-\infty}^{+\infty}\delta\left(v-\frac{k}{Y}\right)$$

$$= \frac{1}{XY}\sum_{\ell,k=-\infty}^{+\infty}\delta\left(u-\frac{\ell}{X}\right)\delta\left(v-\frac{k}{Y}\right) =$$

$$= \frac{1}{XY}\,\text{comb}_{\frac{1}{X}\frac{1}{Y}}(u,v)$$



PROPERTIES OF THE 2-D FOURIER TRANSFORM:

(1) Linearity: $c_1 f_1(x,y) + c_2 f_2(x,y) \xleftrightarrow{\mathcal{F}} c_1 F_1(u,v) + c_2 F_2(u,v)$

(2) SCALING

$$\mathcal{F}\left[f(\alpha x, \beta y)\right] = \frac{1}{|\alpha\beta|}\,F\left(\frac{u}{\alpha},\frac{v}{\beta}\right)$$

Proof: do it for exercise (similar to 1-D)

(3) SHIFT PROPERTY

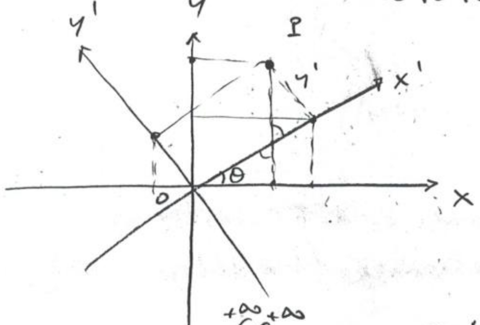
$$\mathcal{F}\left[f(x-x_0, y-y_0)\right] = e^{-j2\pi u x_0}\, e^{-j2\pi v y_0}\, F(u,v)$$

(4) FREQUENCY SHIFT

$$\mathcal{F}\left[e^{j2\pi(u_0 x + v_0 y)} f(x,y)\right] = F(u-u_0, v-v_0)$$

(5) ROTATION PROPERTY

$$\mathrm{rotate}_\theta\left[f(x,y)\right]$$



$$\begin{cases} x = x'\cos\theta - y'\sin\theta = \varphi(x',y') \\ y = x'\sin\theta + y'\cos\theta = \psi(x',y') \end{cases}$$

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

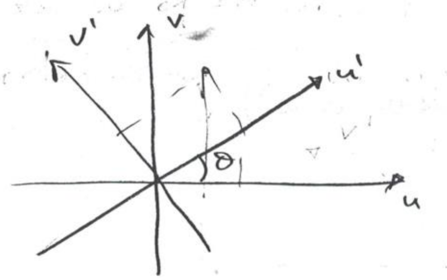
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x',y')\, e^{-j2\pi\left[u(x'\cos\theta - y'\sin\theta) + v(x'\sin\theta + y'\cos\theta)\right]} \left|J(x',y')\right| dx'dy'$$

$$J(x',y') = \begin{vmatrix} \dfrac{\partial\varphi}{\partial x'} & \dfrac{\partial\varphi}{\partial y'} \\ \dfrac{\partial\psi}{\partial x'} & \dfrac{\partial\psi}{\partial y'} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x',y')\, e^{-j2\pi\left[x'(u\cos\theta + v\sin\theta) + y'(-u\sin\theta + v\cos\theta)\right]}\, dx'dy'$$

$= F[u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta]$

= Rotation of θ in (x,y) corresponds to rotation of θ in (u,v)



(5b) COROLLARY OF (5):

$$\mathcal{F}\left[\mathcal{F}\left[f(x,y)\right]\right] = f(-x,-y)$$

__Proof__: Trivial since the inverse 2-D FT differs from the 2-D F.T. by a sign change (rotation $= \pi$).

(6) __CONVOLUTION__

2D convolution:

$$f_1(x,y) * f_2(x,y) \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1(\alpha,\beta)\, f_2(x-\alpha, y-\beta)\, d\alpha\, d\beta$$

$$\mathcal{F}\left[f_1(x,y) * f_2(x,y)\right] = F_1(u,v) \cdot F_2(u,v)$$

(7) __PRODUCT__

$$g(x,y) = f_1(x,y)\, f_2(x,y)$$

PALMERI.17

$$\mathcal{F}\left[g(x,y)\right] = F_1(u,v) * F_2(u,v) \, .$$

PARSEVAL'S THEOREM

$$\iint f_1(x,y)\, f_2^*(x,y)\, dx\, dy = \iint F_1(u,v)\, F_2^*(u,v)\, du\, dv$$

In particolare:

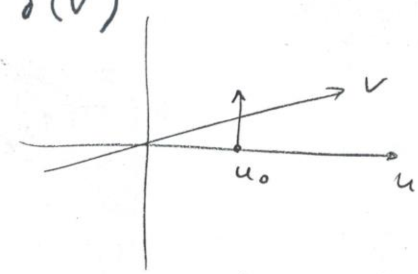
$$\iint \left|f(x,y)\right|^2 dx\, dy = \iint \left|F(u,v)\right|^2 du\, dv$$

OTHER TRANSFORMS

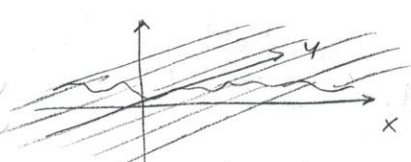
$$\mathcal{F}\left[\delta(x-x_0, y-y_0)\right] = e^{-j2\pi x_0 u} \, e^{-j2\pi y_0 v}$$

$$\mathcal{F}\left[e^{j2\pi u_0 x} \, e^{j2\pi v_0 y}\right] = \delta(u-u_0, v-v_0)$$

$$\mathcal{F}\left[e^{j2\pi u_0 x}\right] = \delta(u-u_0)\, \delta(v)$$

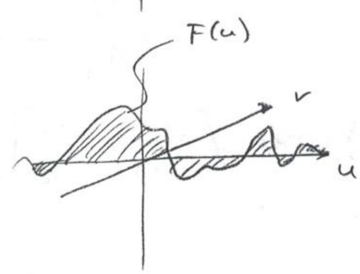


$$\mathcal{F}\left[f(x)\right] = F(u)\, \delta(v)$$



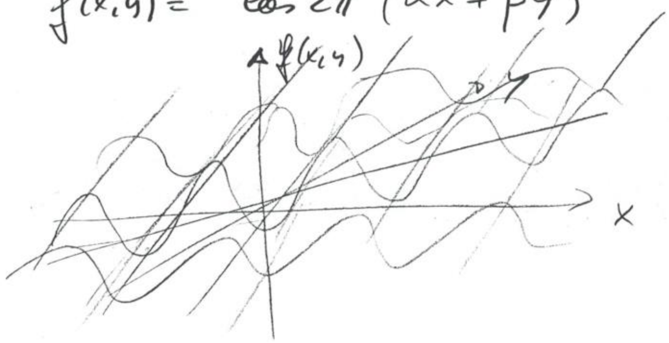
constant in the y direction.

$\Longrightarrow$



EXAMPLES

(a) $f(x,y) = \cos 2\pi\,(\alpha x + \beta y)$



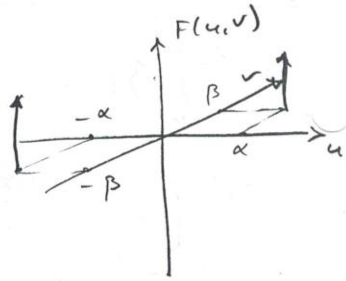
this is not a 2-D cosine! ($\cos 2\pi\alpha x \cdot \cos 2\pi\alpha y$)

$$\mathcal{F}\left[\cos 2\pi(\alpha x + \beta y)\right] = \iint \cos 2\pi(\alpha x+\beta y)\, e^{-j2\pi(ux+vy)}\,dx\,dy$$

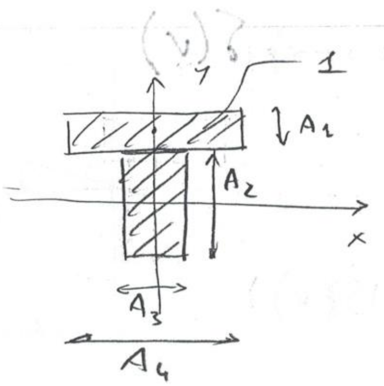
$$= \frac{1}{2}\left[\iint e^{j2\pi(\alpha x+\beta y)}e^{-j2\pi(ux+vy)}\,dx\,dy \;+\right.$$

$$\left. + \iint e^{-j2\pi(\alpha x+\beta y)}e^{-j2\pi(ux+vy)}\,dx\,dy\right]$$

$$= \frac{1}{2}\left[\delta(u-\alpha, v-\beta) + \delta(u+\alpha, v+\beta)\right]$$



(b)

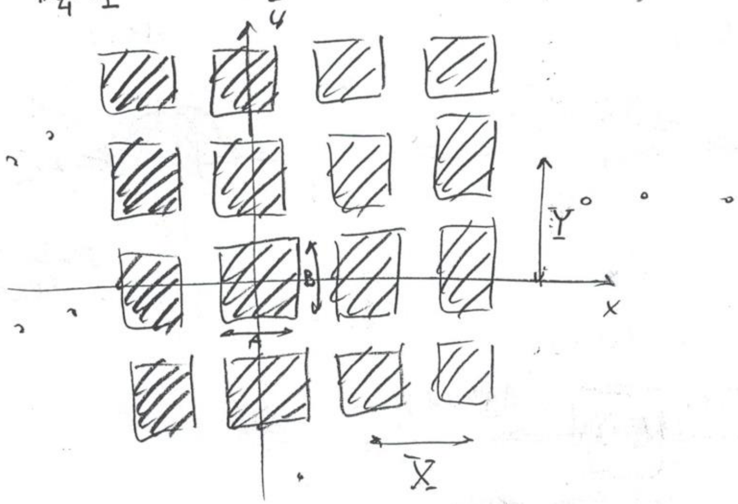


$$f(x,y) = \mathrm{rect}\left[\frac{x}{A_3}, \frac{y}{A_2}\right] + \mathrm{rect}\left[\frac{x}{A_1}, \frac{y - \frac{A_2+A_1}{2}}{A_4}\right]$$

PALMIERI, I.S

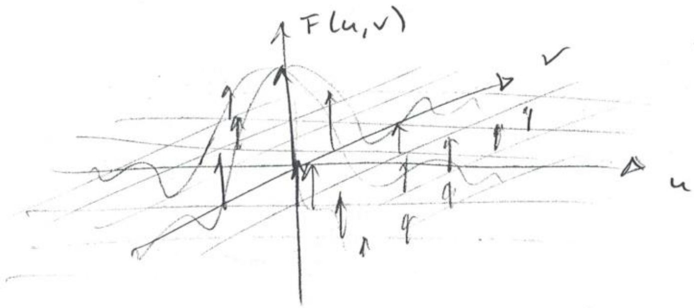
$$F(u,v) = A_3 A_2 \operatorname{sinc} A_3 u \operatorname{sinc} A_2 v + A_4 A_1 \operatorname{sinc} A_1 u \operatorname{sinc} A_4 v \; e^{-j 2\pi \frac{A_2+A_1}{2} v}$$

(e)



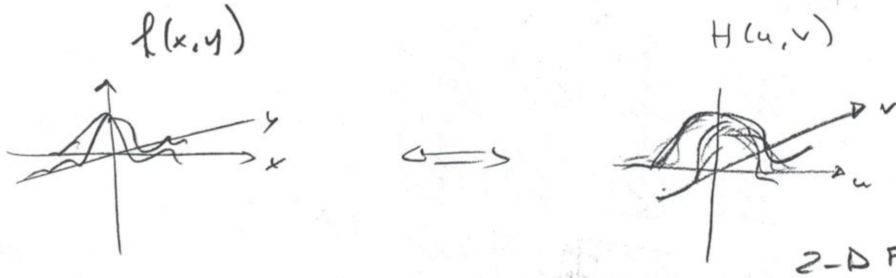
$$f(x,y) = \operatorname{rect}\left[\frac{x}{A}, \frac{y}{B}\right] * \operatorname{comb}_{XY}(x,y)$$

$$F(u,v) = \frac{AB}{XY} \operatorname{sinc} Au \operatorname{sinc} Bv \cdot \operatorname{comb}_{\frac{1}{X}\frac{1}{Y}}(u,v)$$

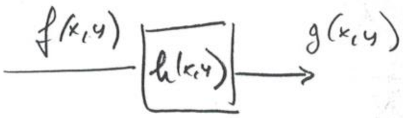


sampled.

A 2-D Filter is then characterized by a 2-D IMPULSE RESPONSE.



2-D FREQUENCY RESPONSE.



2-D convolution.

The behaviour in u will mostly describe filtering of f(x,y) in the x direction, and that in v, filtering in the y direction. We will see how to design 2-D filters.

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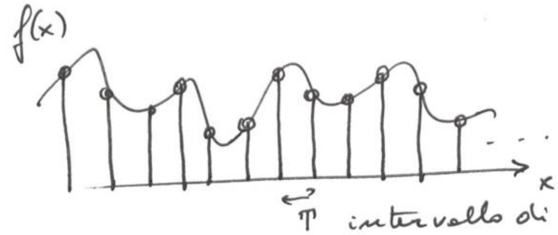
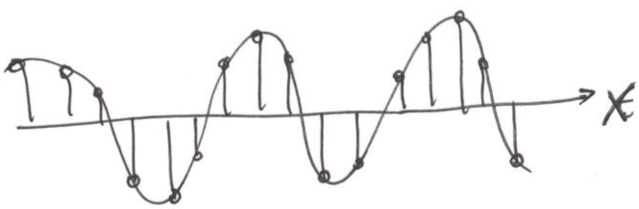
RAPPRESENTAZIONE NUMERICA DELLE IMMAGINI
(NOTE INTUITIVE)

CAMPIONAMENTO MONODIMENSIONALE

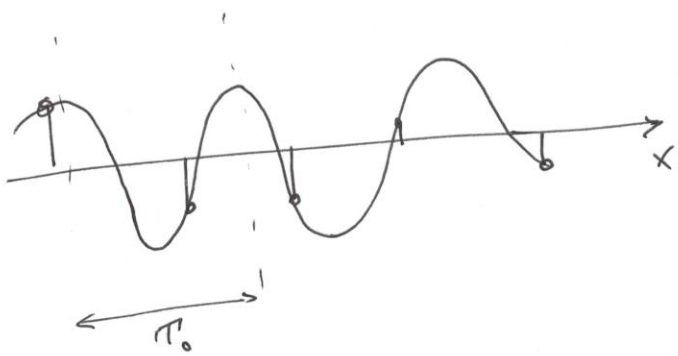


intervallo di campionamento.

$f_s = \frac{1}{T}$ frequenza di campionamento.



segnale ben campionato.



segnale mal campionato.

$\frac{1}{T_0} = f_0$

T. del campionamento:
ottieni almeno due campioni per ciclo

$$T < \frac{T_0}{2}$$

$$\Downarrow$$

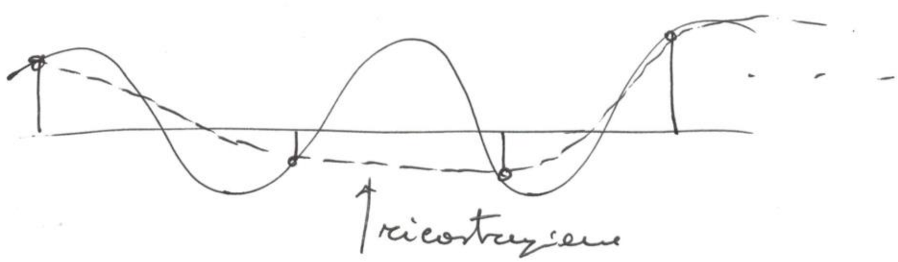
$$\frac{1}{T} > 2\frac{1}{T_0}$$

$$\boxed{f_s > 2 f_0}$$

Se un segnale è scomposto secondo le sue componenti frequenziali delle quali f_{MAX} è quella massima:

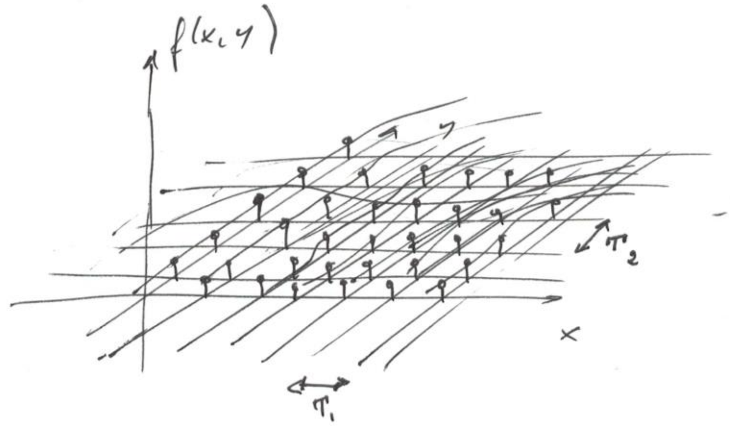
$$\boxed{f_s > 2 f_{MAX}}$$

La ricostruzione che si ottiene da un segnale mal campionato è molto diversa dal segnale originario.



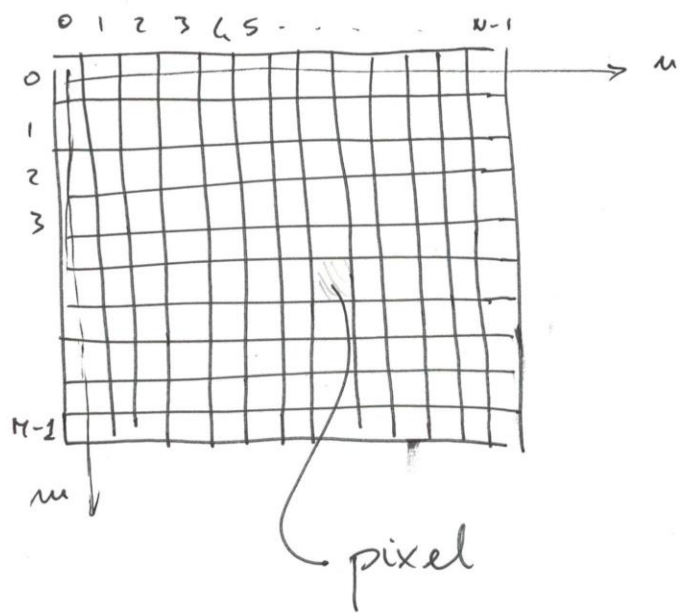
L'errore si chiama "ALIASING"

Ad esempio i segnali audio che hanno frequenza massima $f_{MAX} \simeq 20\,KHZ$ nel CD sono campionati a $\frac{1}{T} = 44\,KHZ$.
I segnali vocali hanno uno spettro molto più limitato $\simeq 4\,KHZ$. Nei sistemi telefonici vengono campionati a 8-10KHZ.



Tipicamente $T_1 = T_2 = T$

Poiché l'immagine non ha una estensione infinita, si ottiene una matrice di numeri.



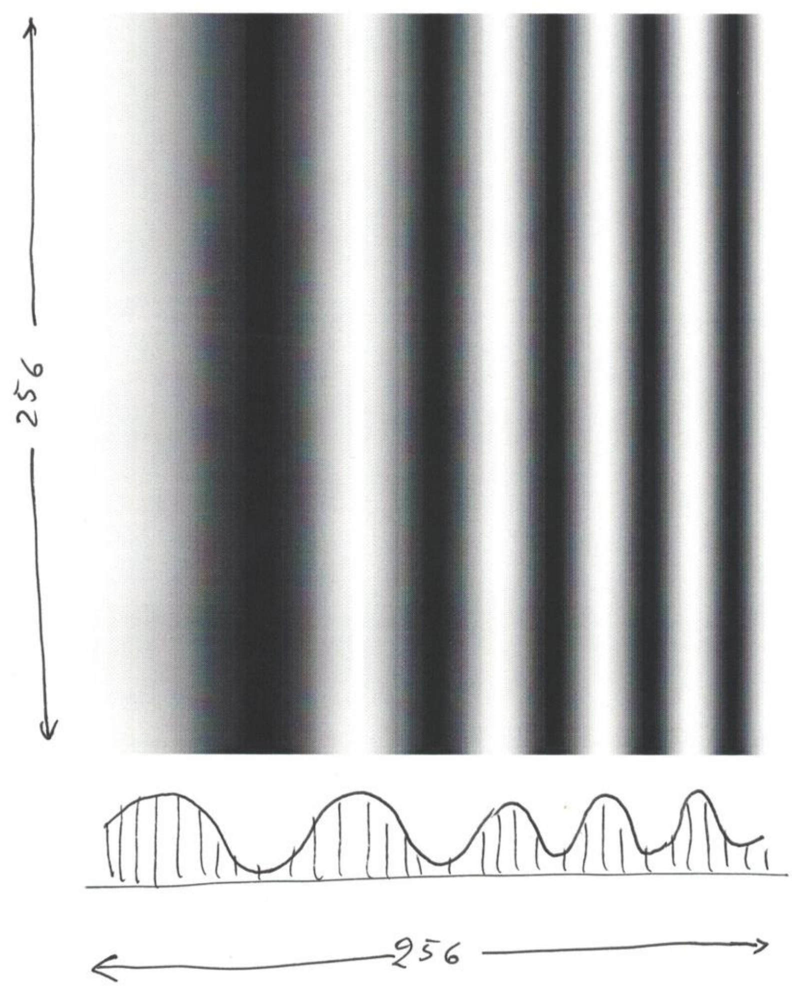
N × M valori.

N, M da ≃100 a 1700

Nella televisione numerica di profilo standard: 576 × 720

" " ad alta risoluzione: 1152 × 1920

I supporti fotografici possono fornire risoluzioni superiori anche di 4÷5 volte.



chirp orizzontale

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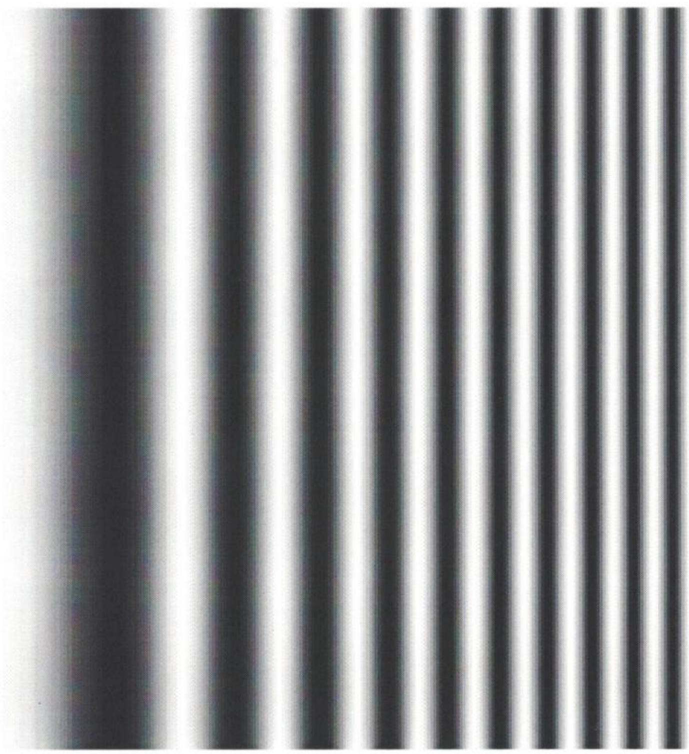

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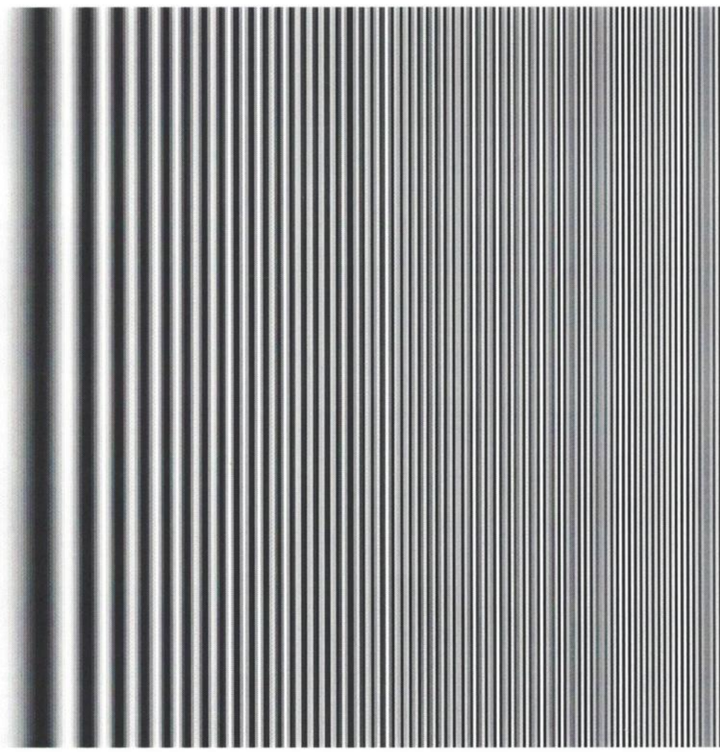
alcuni artefatti!



aliasing

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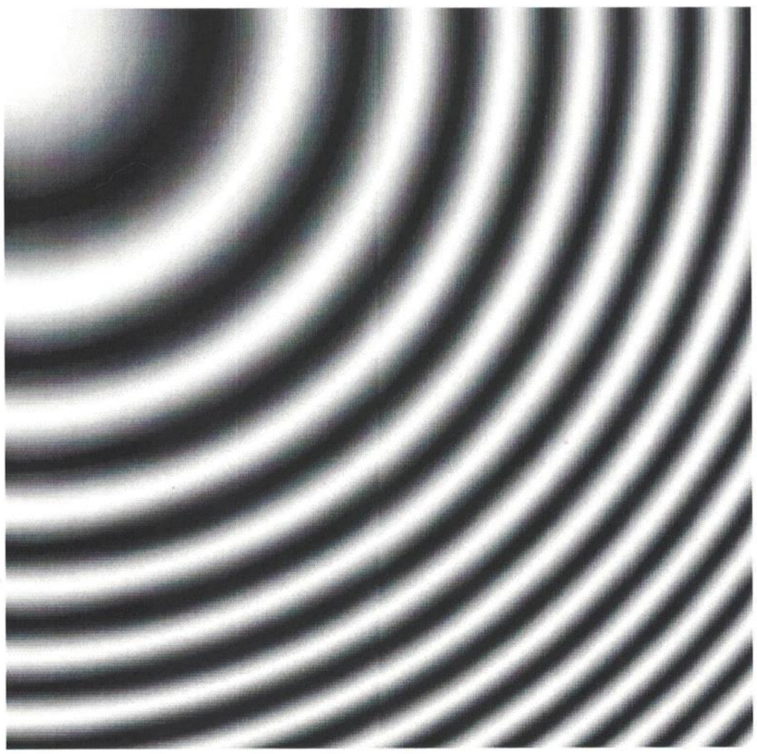

Chirp radiale

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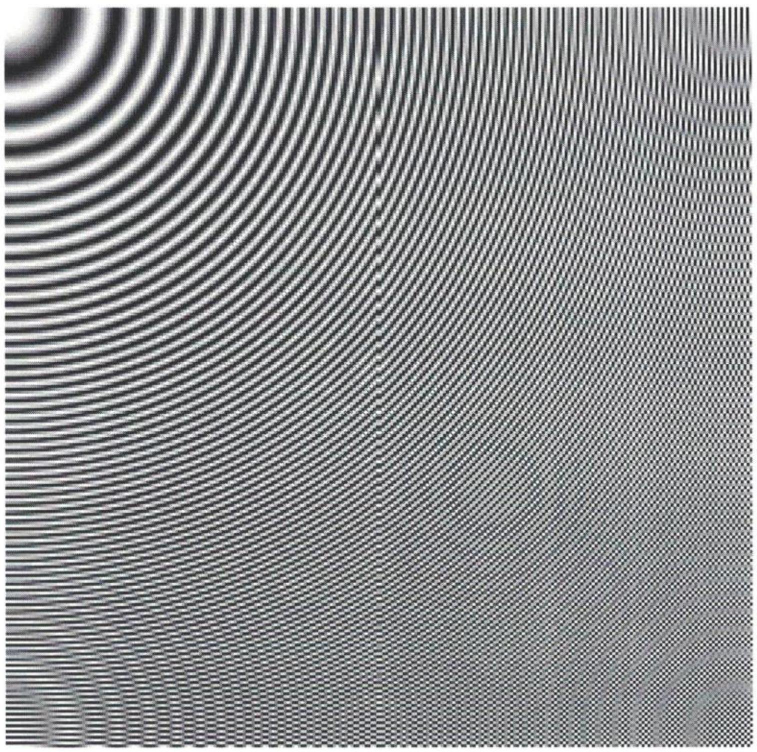

strani artefatti!

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TIPICI FORMATI NUMERICI

La Figura 2.7-4 mostra sinteticamente i rapporti di risoluzione della componente di luminanza nei tre formati CCIR 601, CIF e QCIF.

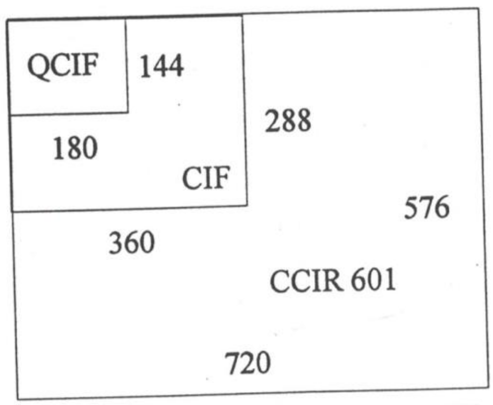

Figura 2.7-4: Risoluzione spaziale nel CCIR 601, nel CIF e nel QCIF.

2-D SAMPLING

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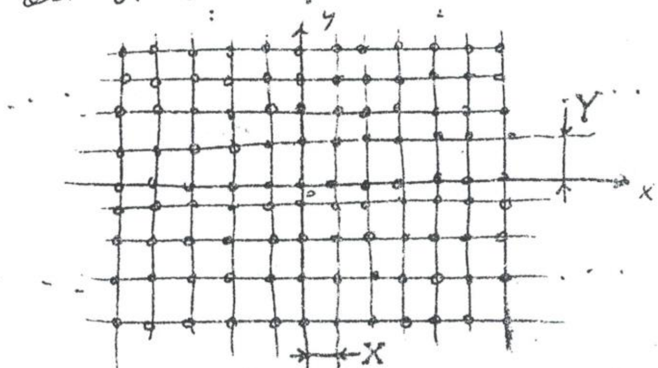
Given a two dimensional function $f(x,y)$ with $(x,y) \in \mathbb{R}^2$ and $f \in \mathbb{R}$, or more generally $f \in \mathbb{C}$, we want to pose the problem of reconstructing $f(x,y)$ from a discrete set of points in the plane (x,y).

RECTANGULAR SAMPLING

Let us assume that the samples of $f(x,y)$ are taken on a rectangular lattice



The sampled function $f_s(x,y)$ can be expressed as

$$f_s(x,y) = f(x,y) \cdot \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \delta(x-nX, y-mY)$$

Taking the Fourier transform we have

$$F_s(u,v) = F(u,v) * \frac{1}{XY} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \delta\left(u-\frac{n}{X}, v-\frac{m}{Y}\right)$$

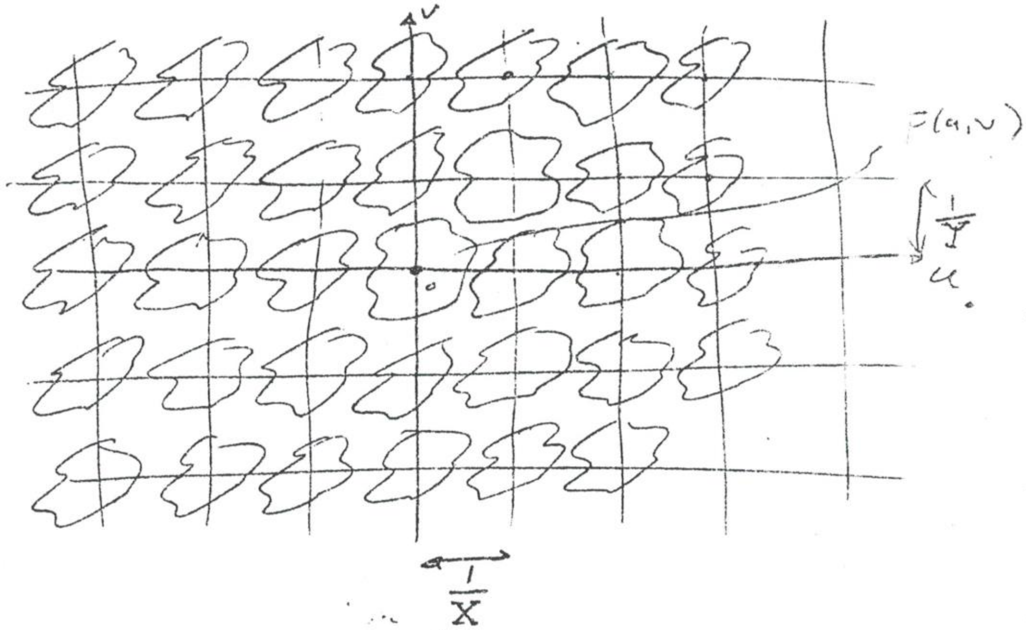
Note: The transform pair

$$\sum_n \sum_m \delta(x-nX, y-mY) \xleftrightarrow{\mathcal{F}} \sum \sum \frac{1}{XY} \delta\left(u-\frac{n}{X}, v-\frac{m}{Y}\right)$$

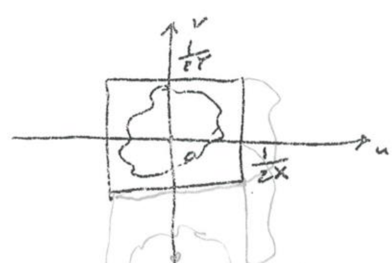
it is simply obtained noting that

$$\sum_n \sum_m \delta(x-nX, y-mY) = \sum_n \delta(x-nX) \sum_m \delta(y-mY)$$

Therefore $F_s(u,v)$ is going to be $F(u,v)$ replied all over the plane (u,v) on a rectangular lattice



Clearly the conditions on the sampling rates $\frac{1}{X}$ in the x direction and $\frac{1}{Y}$ in the y direction will be determined by the region in which $F(u,v)$ is non zero. Therefore for perfect reconstruction through the two-dimensional low-pass filter

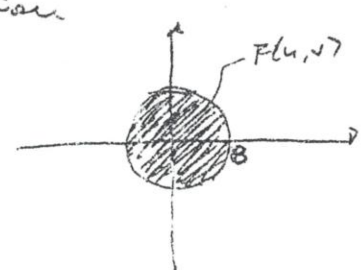


the energy of $F(u,v)$ must be contained inside the above rectangular region.

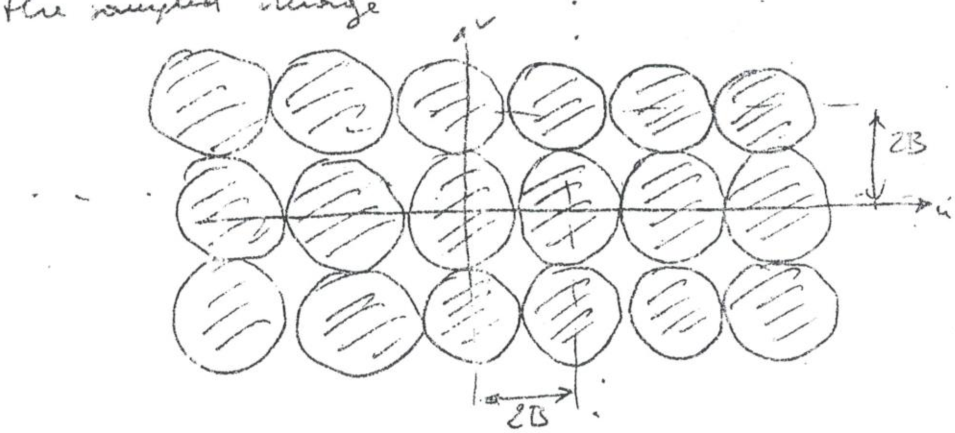
PALMIERI.322

Clearly the rectangular lattice is only one of the many possible ways of defining a lattice of points to sample a two-dimensional function (image).

In fact, if the spectrum $F(u,v)$ is limited to a circular region



the rectangular sampling would give a spectrum for the sampled image



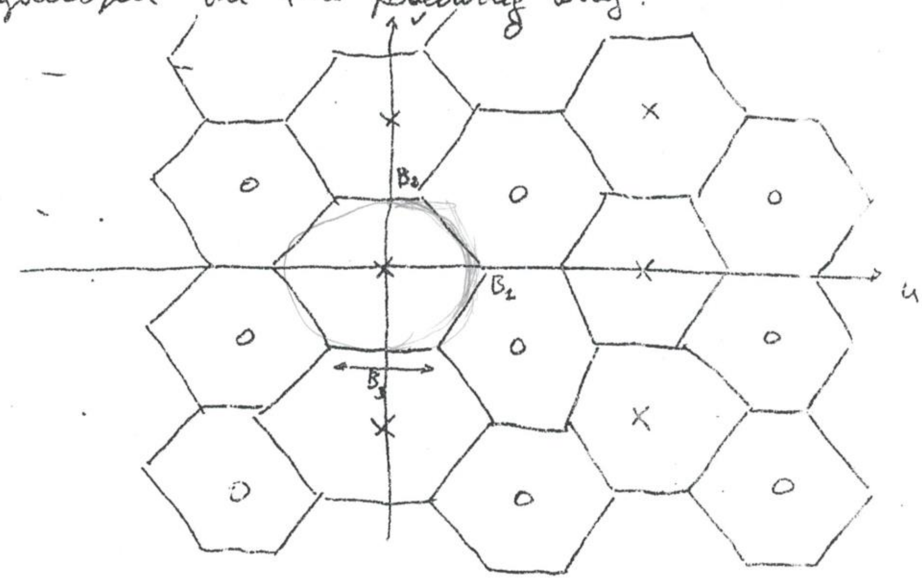
The conditions on the sampling in the (x,y) plane are

$$X \le \frac{1}{2B}, \quad Y \le \frac{1}{2B}.$$

In this particular example, and in general in any cases of images having spectra that exhibit circular symmetry a popular way of obtaining more efficient sampling is to use hexagonal sampling

HEXAGONAL SAMPLING

The lattice of points, where the replicas of the "baseband" spectrum $F(u,v)$ are located, can be arranged in the following way:



The spectrum of the sampled image will be

$$F_S(u,v) = G(u,v) * D(u,v)$$

where the function $D(u,v)$ is the superposition of the Dirac functions located on the hexagonal lattice. $D(u,v)$ can be expressed as the superposition of two rectangular lattices one shifted with respect to the other. Namely

$$D(u,v) = \sum_{K_1=-\infty}^{+\infty} \sum_{K_2=-\infty}^{+\infty} \Big[\delta\big(u - K_1(2B_1+B_3),\, v - 2K_2 B_2\big) + \delta\Big(u - K_1(2B_1+B_3) - \frac{2B_1+B_3}{2},\, v - 2B_2K_2 - B_2\Big)\Big]$$

let us rewrite it as

$$D(u,v) = \sum_{K_1} \sum_{K_2} \Big[\delta\Big(u - 2K_1\Big(\frac{2B_1+B_3}{2}\Big),\, v - 2K_2B_2\Big) + \delta\Big(u - (2K_1+1)\Big(\frac{2B_1+B_3}{2}\Big),\, v - (2K_2+1)B_2\Big)\Big]$$

let $T_1 = \frac{2}{2B_1 + B_3}$ $\qquad T_2 = \frac{1}{2B_2}$

$$D(u,v) = \sum_{k_1}\sum_{k_2}\left[\delta\left(u - \frac{2k_1}{T_1}, v - \frac{k_2}{T_2}\right) + \right.$$

$$\left. + \delta\left(u - \frac{(2k_1+1)}{T_1}, v - \frac{(2k_2+1)}{2T_2}\right)\right] =$$

$$= \sum_{k_1}\sum_{k_2}\left[\delta\left(u - \frac{2k_1}{T_1}, v - \frac{k_2}{T_2}\right) + \right.$$

$$\left. + \delta\left(u - \frac{2k_1}{T_1} - \frac{1}{T_1}, v - \frac{k_2}{T_2} - \frac{1}{2T_2}\right)\right]$$

The sampling lattice in the (x,y) plane is going to be given by

$$d(x,y) = \frac{T_1 T_2}{2}\sum_{k_1}\sum_{k_2}\delta\left(x - \frac{k_1 T_1}{2}, y - k_2 T_2\right)\cdot$$

$$\cdot\left[1 + e^{j\frac{2\pi x}{T_1}} e^{j\frac{2\pi y}{2T_2}}\right]_{x = \frac{k_1 T_1}{2};\, y = k_2 T_2} =$$

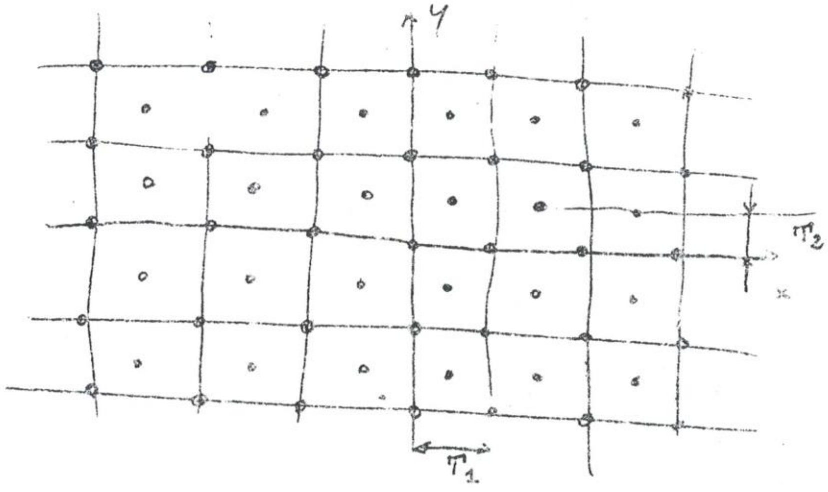
$$= \frac{T_1 T_2}{2}\sum_{k_1}\sum_{k_2}\left(1 + (-1)^{k_1+k_2}\right)\delta\left(x - \frac{k_1 T_1}{2}, y - k_2 T_2\right)$$

$$f_S(x,y) = f(x,y)\cdot d(x,y)$$

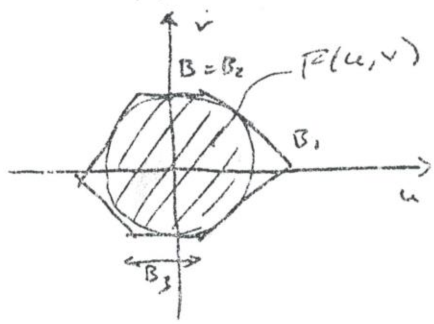
the lattice in the (x,y) plane is :

PALMIERI. I25

6

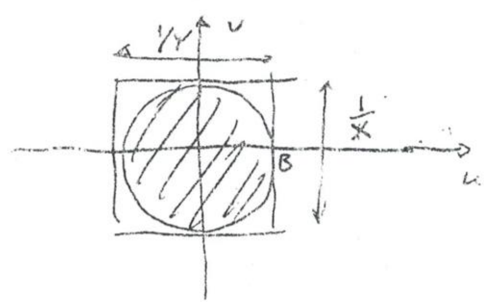


To see what is the advantage of using hexagonal sampling instead of rectangular sampling, let us consider a circular bandlimited function



$B_2 = B$

$B_1 = B_3 = \frac{2}{\sqrt{3}} B_2$



PALMIERI - I26

The conditions for rectangular sampling are

$$\begin{cases} X \le \frac{1}{2B} \\ Y \le \frac{1}{2B} \end{cases}$$

Instead with hexagonal sampling

$$T_1 = \frac{2}{2B_1 + B_3} = \frac{2}{3\frac{2}{\sqrt{3}}B} = \frac{1}{\sqrt{3}\,B} = \frac{1}{1.74 \cdot B}$$

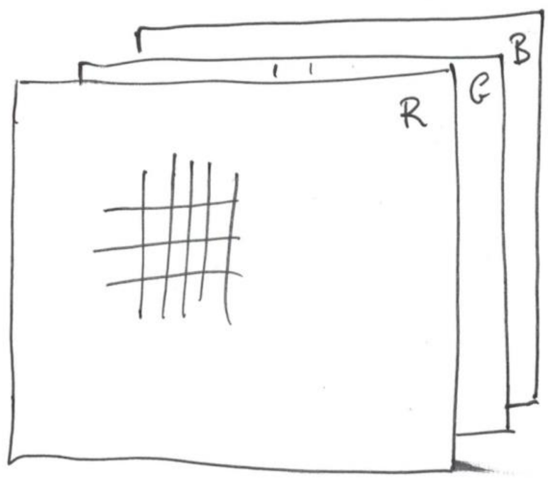
$$T_2 = \frac{1}{2B_2} = \frac{1}{2B}$$

$$\begin{cases} T_1 \le \frac{1}{1.74 \cdot B} \\ T_2 \le \frac{1}{2B} \end{cases}$$

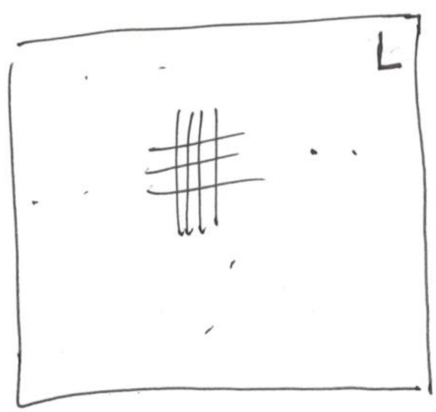
T_1 is about 14% larger than X.

Therefore in the X direction we can use a smaller number of points. We use a smaller number of points per unit of area.

LUMINANZA E COMPONENTI DI CROMINANZA



Come definire una immagine di luminanza?
(in bianco e nero)

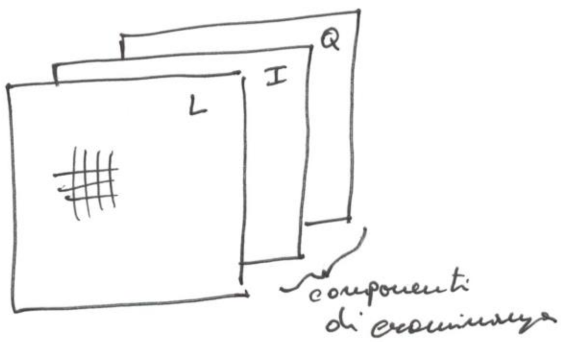


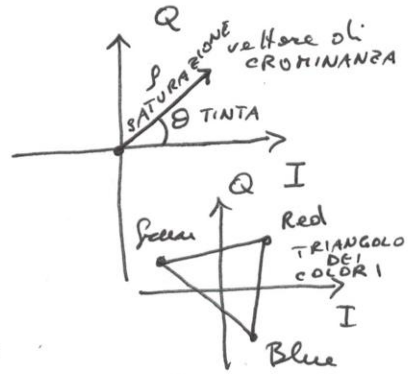
$$L = 0.299\,R + 0.587\,G + 0.114\,B$$

Formula sperimentale che porta ad una immagine di luminanza un pò più "naturale"

Come aggiungere alla immagine di luminanza le informazioni del colore?

$$\begin{cases} I = 0.596\,R - 0.274\,G - 0.322\,B \\ Q = 0.211\,R - 0.523\,G + 0.312\,B \end{cases} \quad \begin{pmatrix}\text{sistema} \\ \text{NTSC}\end{pmatrix}$$





ovviamente è possibile in maniera invertibile passare da una rappresentazione all'altra

$$\begin{bmatrix} L \\ I \\ Q \end{bmatrix} = \begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} & & \\ & T^{-1} & \\ & & \end{bmatrix} \begin{bmatrix} L \\ I \\ Q \end{bmatrix}$$