# TRASFORMAZIONI DA FILTRO PASSA-BASSO ANALOGICHE E NUMERICHE

Lezioni di Telecomunicazioni

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# TRASFORMAZIONI (filtri analogici)

Filtro Analogico Passa Basso $H_{LP}(s)$ $\longrightarrow$ Filtro Analogico $H_a(s)$
- passa basso
- passa alto
- passa banda
- a banda oscura.

Ottenuto un filtro passa-basso dalle specifiche di progetto, per passare facilmente ad un filtro di natura diversa esistono delle semplici formule di trasformazione.
In particolare il filtro analogico finale viene ottenuto per semplice sostituzione.

$$H_a(s') = H_{LP}(s)\Big|_{s=g(s')}$$

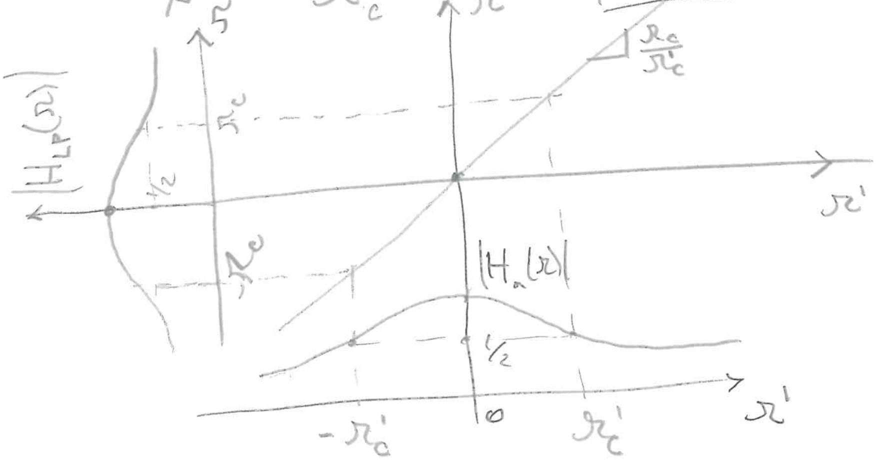
Esaminiamo le 4 principali formule di trasformazione

PASSA-BASSO freq. di taglio $\Omega_c$ $\Longrightarrow$ PASSA-BASSO freq. di taglio $\Omega'_c$

$$\boxed{s = \frac{\Omega_c}{\Omega'_c}\, s'}$$

La trasformazione è la più semplice e realizza una semplice riscalatura del piano s. In particolare l'asse $j\Omega$ si trasforma in maniera invertibile nell'asse $j\Omega'$

$$j\Omega = \frac{\Omega_c}{\Omega'_c}\, j\Omega' \; ; \quad \boxed{\Omega = \frac{\Omega_c}{\Omega'_c}\,\Omega'} \quad (*)$$



È evidente come in questo caso sia semplice verificare che se $H_{LP}(s)$ è stabile (poli a parte reale negativa) anche $H_a(s)$ è stabile: Il piano sinistro di $s$ si mappa direttamente nel piano sinistro di $s'$.
Se il filtro passa-basso è stato progettato con una maschera con specifiche di attenuazione e $\Omega_p$ e $\Omega_s$, le nuove $\Omega_p'$ e $\Omega_s'$ possono essere ottenute direttamente dalla formula (*): $\Omega' = \frac{\Omega_c'}{\Omega_c}\Omega$

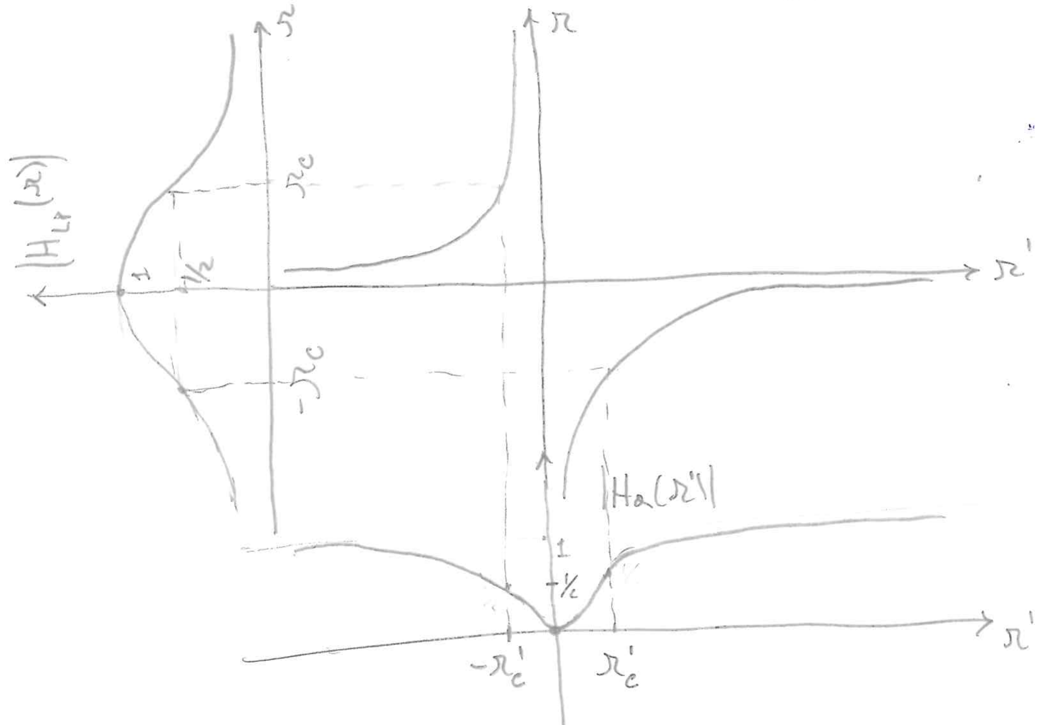
| | | |
|---|---|---|
| PASSA-BASSO freq. di taglio $\Omega_c$ | $\Longrightarrow$ | PASSA ALTO freq. di taglio $\Omega_c'$ |

$$\boxed{S = \frac{\Omega_c \Omega_c'}{s'}}$$

La trasformazione è ora non lineare. Notiamo innanzitutto che l'asse $j\Omega'$ è diretta trasformazione di $j\Omega$. Infatti per $s' = j\Omega'$

$$jS = \frac{\Omega_c\Omega_c'}{j\Omega'} = -j\frac{\Omega_c\Omega_c'}{\Omega'} = j\Omega$$

La trasformazione delle frequenze è $\boxed{\Omega = -\frac{\Omega_c\Omega_c'}{\Omega'}}$ (**)




Qui la banda passante si ritrova a destra di $\Omega_c'$ e la banda oscura al di sotto.

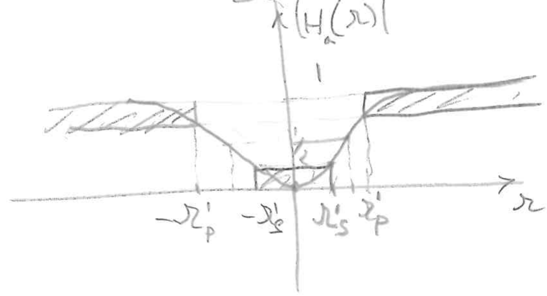
Per verificare la conservazione della stabilità basta scrivere 

$$s' = \frac{\Omega_c \Omega_c'}{s} = \frac{\Omega_c \Omega_c'}{\alpha + j\Omega} = \frac{\Omega_c \Omega_c' (\alpha - j\Omega)}{\alpha^2 + \Omega^2}$$

$$\alpha' + j\Omega' = \underbrace{\frac{\Omega_c \Omega_c' \alpha}{\alpha^2 + \Omega^2}}_{\alpha'} - \underbrace{\frac{\Omega_c \Omega_c' \Omega}{\alpha^2 + \Omega^2}}_{\Omega'}$$

($\alpha'$ e $\alpha$ hanno lo stesso segno, quindi il piano sinistro di s si mappa nel piano sinistro di s')

Le specifiche del filtro passa-basso in termini di $\Omega_p$ e $\Omega_s$ vanno tradotte in termini di $\Omega_p'$ e $\Omega_s'$ come in figura da formula (**): $\Omega' = -\frac{\Omega_c \Omega_c'}{\Omega}$

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PASSA-BASSO fre. di taglio $\Omega_c$ $\Longrightarrow$ PASSA-BANDA $\Omega_e, \Omega_u$

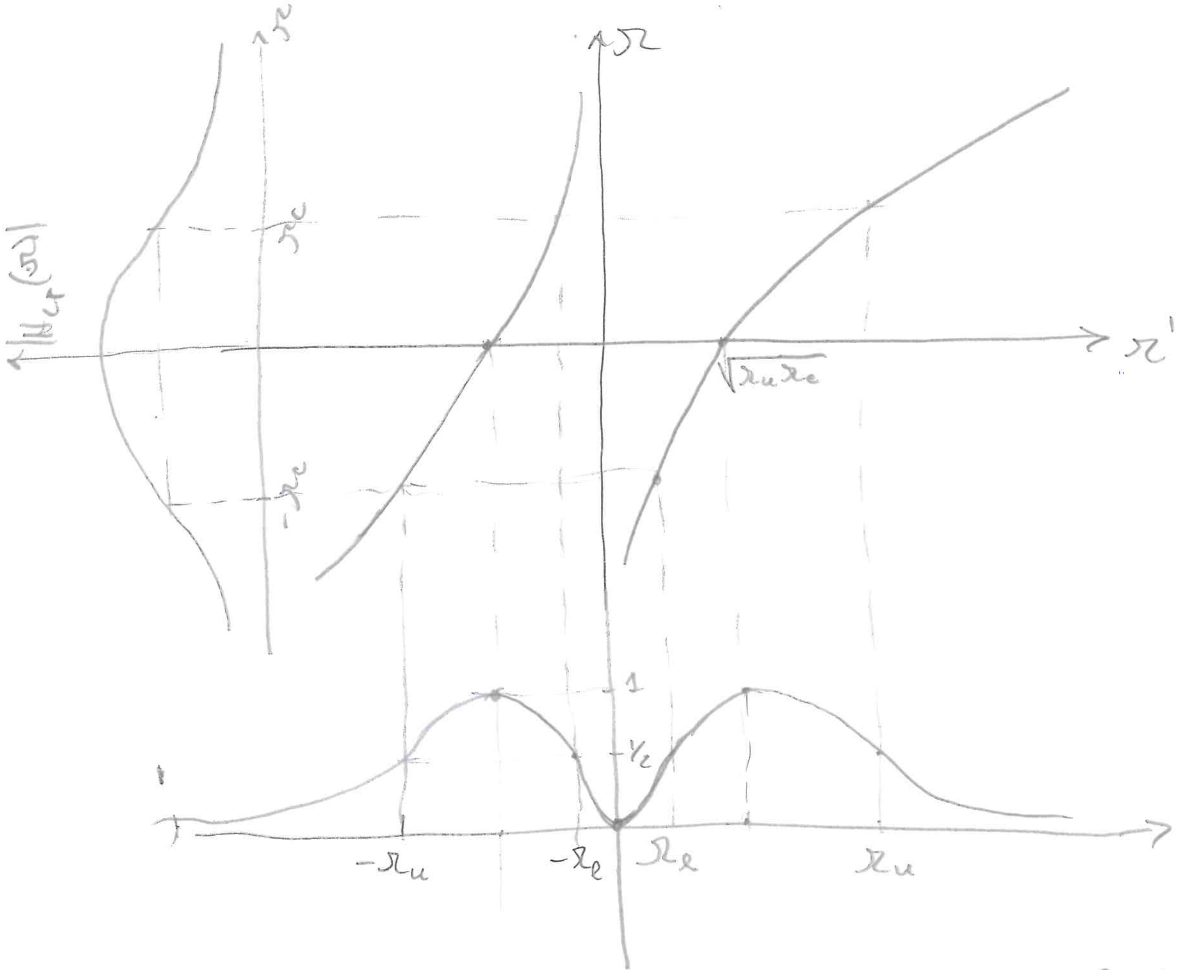
$$\boxed{s = \Omega_c \frac{s'^2 + \Omega_e \Omega_u}{s'(\Omega_u - \Omega_e)}}$$

La prima cosa da notare è che per $s' = j\Omega'$ otteniamo

$$s = \Omega_c \frac{(j\Omega')^2 + \Omega_e \Omega_u}{j\Omega'(\Omega_u - \Omega_e)} = -\Omega_c j \Omega' \frac{-\Omega'^2 + \Omega_e \Omega_u}{\Omega_u - \Omega_e} = j\Omega$$

Quindi la formula di trasformazione delle frequenze da $\Omega'$ a $\Omega$ è

$$\boxed{\Omega = \frac{\Omega_c}{\Omega_u - \Omega_e} \frac{\Omega'^2 - \Omega_e \Omega_u}{\Omega'}} \quad (***)$$




La banda passante si mappa tra $\Omega_e$ e $\Omega_u$ e la banda oscura tra $0$ e $\Omega_e$ e tra $\Omega_u$ e infinito.

Per verificare la stabilità basta analizzare la formula di trasformazione rendendo evidente la parte reale, ovvero per $s' = \alpha' + j\Omega'$ otteniamo

$$s = \frac{\Omega_c}{\Omega_u - \Omega_e}\left[s' + \frac{\Omega_e \Omega_u}{s'}\right] = \frac{\Omega_c}{\Omega_u - \Omega_e}\left[(\alpha' + j\Omega') + \Omega_e \Omega_u \frac{(\alpha' - j\Omega')}{\alpha'^2 + \Omega'^2}\right] =$$

$$= \frac{\Omega_c}{\Omega_u - \Omega_e}\left\{\underbrace{\left[1 + \frac{\Omega_e \Omega_u}{\alpha'^2 + \Omega'^2}\right]\alpha'}_{Re} + j\underbrace{\left[1 + \frac{\Omega_e \Omega_u}{\alpha'^2 + \Omega'^2}\right]\Omega'}_{Im}\right\}$$

Quindi $\alpha'$ conserva il segno di $\alpha$, ovvero se $H_{LP}(s)$ è stabile, anche $H_a(s)$ è stabile.

Per la trasformazione delle specifiche, notiamo che ogni frequenza in $\Omega$ corrisponde a due frequenze in $\Omega'$. In particolare dal grafico si nota che $\Omega_c$ si trasforma in $\Omega_u$ e $-\Omega_e$ e $-\Omega_c$ in $\Omega_e$ e $-\Omega_u$.

Questo si vede immediatamente invertendo la (***)



$$r_c r'^2 - (r_u - r_e) r r' - r_c r_e r_u = 0$$

$$r' = \frac{r(r_u - r_e) \pm \sqrt{(r_u - r_e)^2 r^2 + 4 r_c^2 r_e r_u}}{2 r_c} = \begin{cases} h_1(r) \\ h_2(r) \end{cases}$$

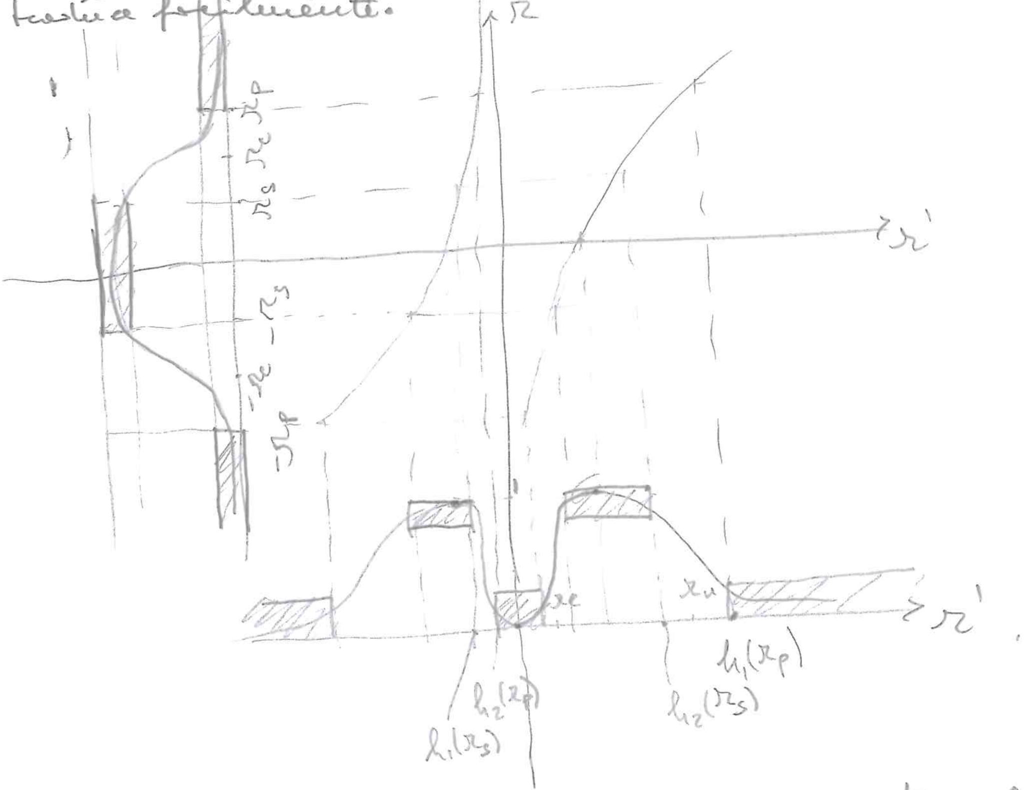
Infatti per $r = r_c$

$$r' = \frac{r_c(r_u - r_e) \pm \sqrt{(r_u^2 + r_e^2 - 2 r_u r_e + 4 r_e r_u) r_c^2}}{2 r_c}$$

$$= \frac{\cancel{r_c}(r_u - r_e) \pm (r_u + r_e)\cancel{r_c}}{2\cancel{r_c}} = \begin{cases} r_u \\ -r_e \end{cases}$$

finalmente per $r = -r_e$ $r' = \begin{cases} r_e \\ -r_u \end{cases}$

Usando le soluzioni $h_1(r)$ e $h_2(r)$ la maschera delle specifiche nel passa-basso si traduce facilmente.




le caratteristiche nella banda passante del passa-basso si traducono in quelle della banda passante del passa-banda. Quelle della banda oscura del passa-basso nelle caratteristiche delle due bande oscure del passa-banda.

—— o ——

PASSA-BASSO freq. di taglio $\Omega_c$ $\Longrightarrow$ FILTRO A BANDA OSCURA in $[\Omega_e\ \Omega_u]$.

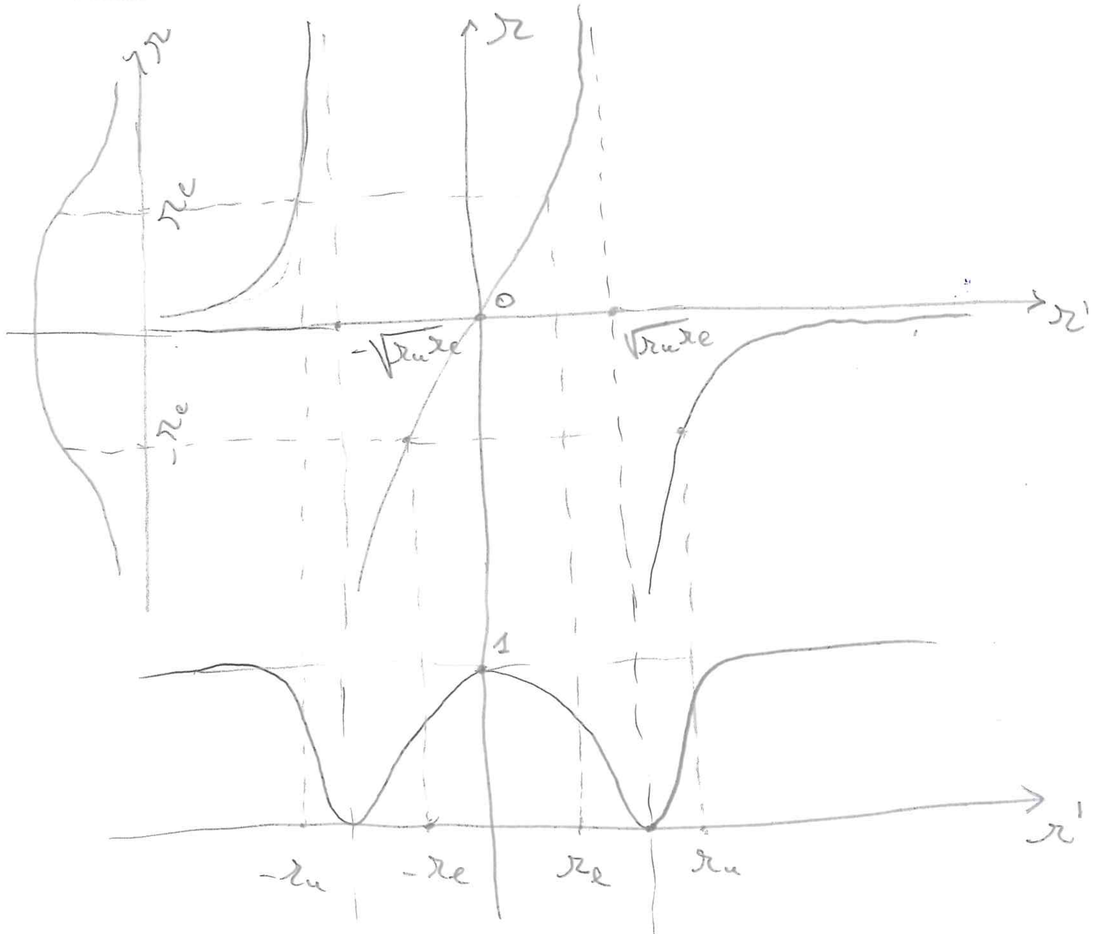
$$\boxed{S = \Omega_c \frac{s'(\Omega_u-\Omega_e)}{s'^2+\Omega_u\Omega_e}}$$

Anche qui per $s'=j\Omega'$ otteniamo

$$S = \Omega_c \frac{j\Omega'(\Omega_u-\Omega_e)}{-\Omega'^2+\Omega_u\Omega_e}$$ puramente immaginario.

Quindi anche qui l'asse $j\Omega$ e l'asse $j\Omega'$ sono in relazione diretta e la formula della trasformazione delle frequenze è

$$\boxed{\Omega = \Omega_c \frac{\Omega'(\Omega_u-\Omega_e)}{-\Omega'^2+\Omega_u\Omega_e}} \quad (***)$$



Per la conservazione della stabilità esaminiamo più nel dettaglio la formula di trasformazione. Per $s' = \alpha' + j\Omega'$ TA7

$$s = \Omega_c \frac{(\alpha'+j\Omega')(\Omega_u-\Omega_e)}{(\alpha'+j\Omega')^2+\Omega_u\Omega_e} =$$

$$= \Omega_c(\Omega_u-\Omega_e)\frac{\alpha'+j\Omega'}{\alpha'^2-\Omega'^2+2j\alpha'\Omega'+\Omega_u\Omega_e} = \frac{\Omega_c(\Omega_u-\Omega_e)(\alpha'+j\Omega')(\alpha'^2-\Omega'^2+\Omega_u\Omega_e-2j\alpha'\Omega')}{|\cdot|^2}$$

$$= \frac{\Omega_c(\Omega_u-\Omega_e)}{|\cdot|^2}\left[\underbrace{\alpha'^3-\cancel{\alpha'\Omega'^2}+\Omega_u\Omega_e\alpha'+\cancel{\Omega'^2\alpha'}}_{Re} + j\left(\Omega'(\alpha'^2-\Omega'^2+\Omega_u\Omega_e)-\alpha'^2\Omega'\right)\right]$$

$$\alpha'(\alpha'^2+\Omega_u\Omega_e)$$

La parte reale conserva il segno quindi il semipiano sinistro di s' e di s sono corrispondenti. Quindi un filtro passa-basso stabile resta stabile.

Per trasformare le specifiche del passa-basso nelle specifiche del filtro a banda oscura notiamo che ad ogni frequenza del passa-basso corrispondono due valori di $\Omega'$. Questo si vede invertendo la formula di trasformazione delle frequenze (****).

$$-\Omega'^2\Omega+\Omega_u\Omega_e\Omega = \Omega_c\Omega'(\Omega_u-\Omega_e)$$

$$\Omega\Omega'^2+\Omega_c(\Omega_u-\Omega_e)\Omega'-\Omega_u\Omega_e\Omega = 0$$

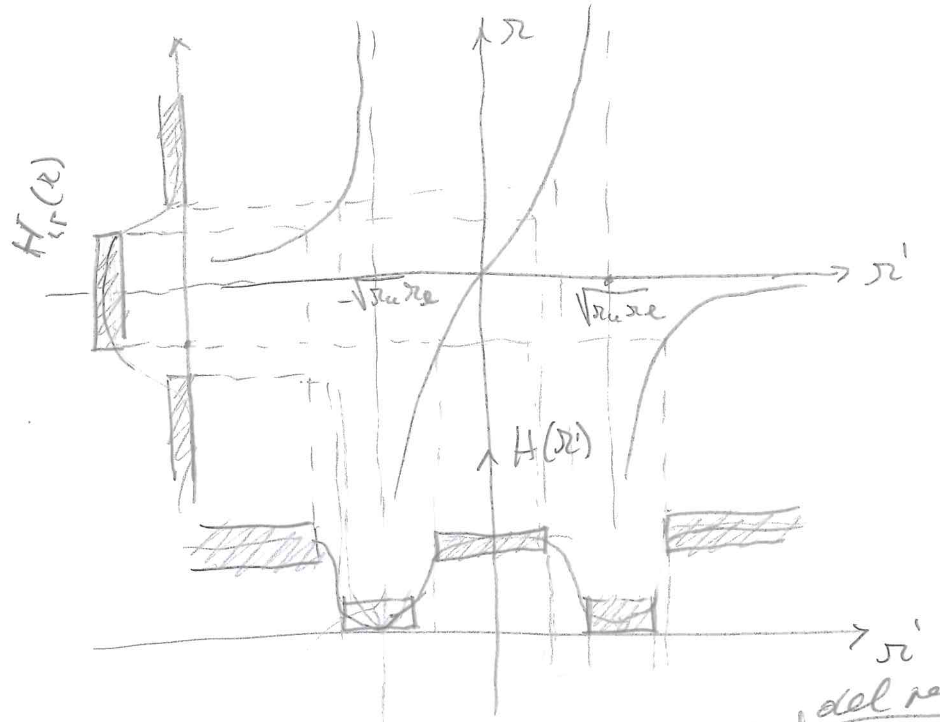
$$\boxed{\Omega' = \frac{-\Omega_c(\Omega_u-\Omega_e)\pm\sqrt{\Omega_c^2(\Omega_u-\Omega_e)^2+4\Omega_u\Omega_e\Omega^2}}{2\Omega}}$$

Infatti per $\Omega = \Omega_c$

$$\Omega' = \frac{-\Omega_c(\Omega_u-\Omega_e)\pm\sqrt{\Omega_c^2\Omega_u^2+\Omega_c^2\Omega_e^2-2\Omega_c^2\Omega_u\Omega_e+4\Omega_u\Omega_e\Omega_c^2}}{2\Omega_c}$$

$$= \frac{-\Omega_c(\Omega_u-\Omega_e)\pm\Omega_c(\Omega_u+\Omega_e)}{2\Omega_c} = \begin{cases}\Omega_e\\ -\Omega_u\end{cases}$$

Analogamente per $\Omega = -\Omega_c$ $\quad \Omega' = \begin{cases}\Omega_u\\ -\Omega_e\end{cases}$



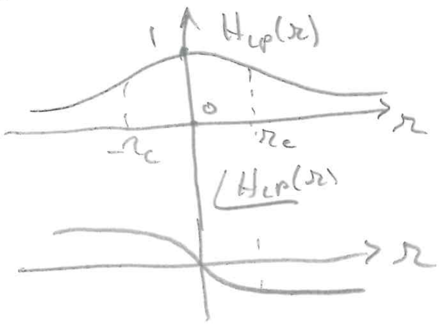

Le specifiche della banda passante del passa-basso saranno le specifiche delle 2 bande passanti (prima e dopo la banda oscura). Le specifiche sulla banda oscura del passa-basso saranno le caratteristiche della banda oscura del filtro finale.

Esempio

Passa-basso (Butterworth) $\longrightarrow$ PASSA BANDA $\Omega_e\,\Omega_u$

$$H_{LP}(s) = \frac{\Omega_c}{s+\Omega_c} \quad ; \quad H_{LP}(\Omega) = \frac{\Omega_c}{j\Omega+\Omega_c} = \frac{1}{1+j\frac{\Omega}{\Omega_c}}$$

$$|H_{LP}(\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^2}} \quad . \quad \angle H_{LP}(\Omega) = -\operatorname{tg}^{-1}\frac{\Omega}{\Omega_c}$$




$f_c = 1000$ Hz $\quad \Omega_c = 2\cdot\pi\cdot 1000 = 6.28\cdot 10^3$ rad/sec.

$(f_e, f_u) = (5,6)$ KHz. $(\Omega_e\,\Omega_u) = (2\pi\cdot 5000,\ 2\pi\, 6000)$
$= (15710,\ 18852)$ rad/sec.

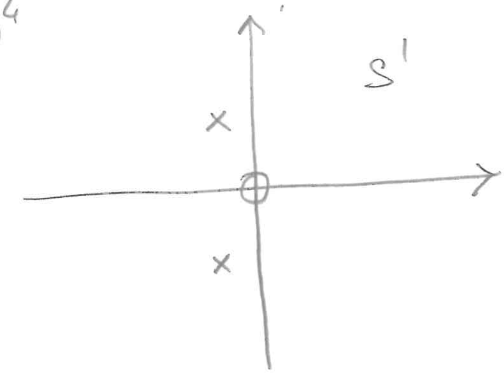
$$H'(s') = H_{LP}(s)\Big|_{s=\Omega_c\frac{s'^2+\Omega_e\Omega_u}{s'(\Omega_u-\Omega_e)}} = \frac{\cancel{\Omega_c}}{\cancel{\Omega_c}\frac{s'^2+\Omega_e\Omega_u}{s'(\Omega_u-\Omega_e)}+\cancel{\Omega_c}}$$

$$= \frac{s'(\Omega_u-\Omega_e)}{s'^2+\Omega_e\Omega_u+s'(\Omega_u-\Omega_e)} = (\Omega_u-\Omega_e)\frac{s'}{s'^2+(\Omega_u-\Omega_e)s'+\Omega_e\Omega_u}$$

poli: $$p_{12} = \frac{-(\Omega_u-\Omega_e)\pm\sqrt{(\Omega_u-\Omega_e)^2-4\Omega_e\Omega_u}}{2} = \frac{\Omega_u-\Omega_e}{2}\left(-1\pm\sqrt{1-\frac{4\Omega_e\Omega_u}{(\Omega_u-\Omega_e)^2}}\right)$$

$$= \begin{cases}(-0.3142 + j\,3.4211)\cdot 10^4 \\ (-0.3142 - j\,3.4211)\cdot 10^4\end{cases}$$

$z_1 = 0$ uno zero nell'origine




$\downarrow$ plot con freqz.

TA 10

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freqs([omu-oml 0],[1 omu-oml omu*oml])
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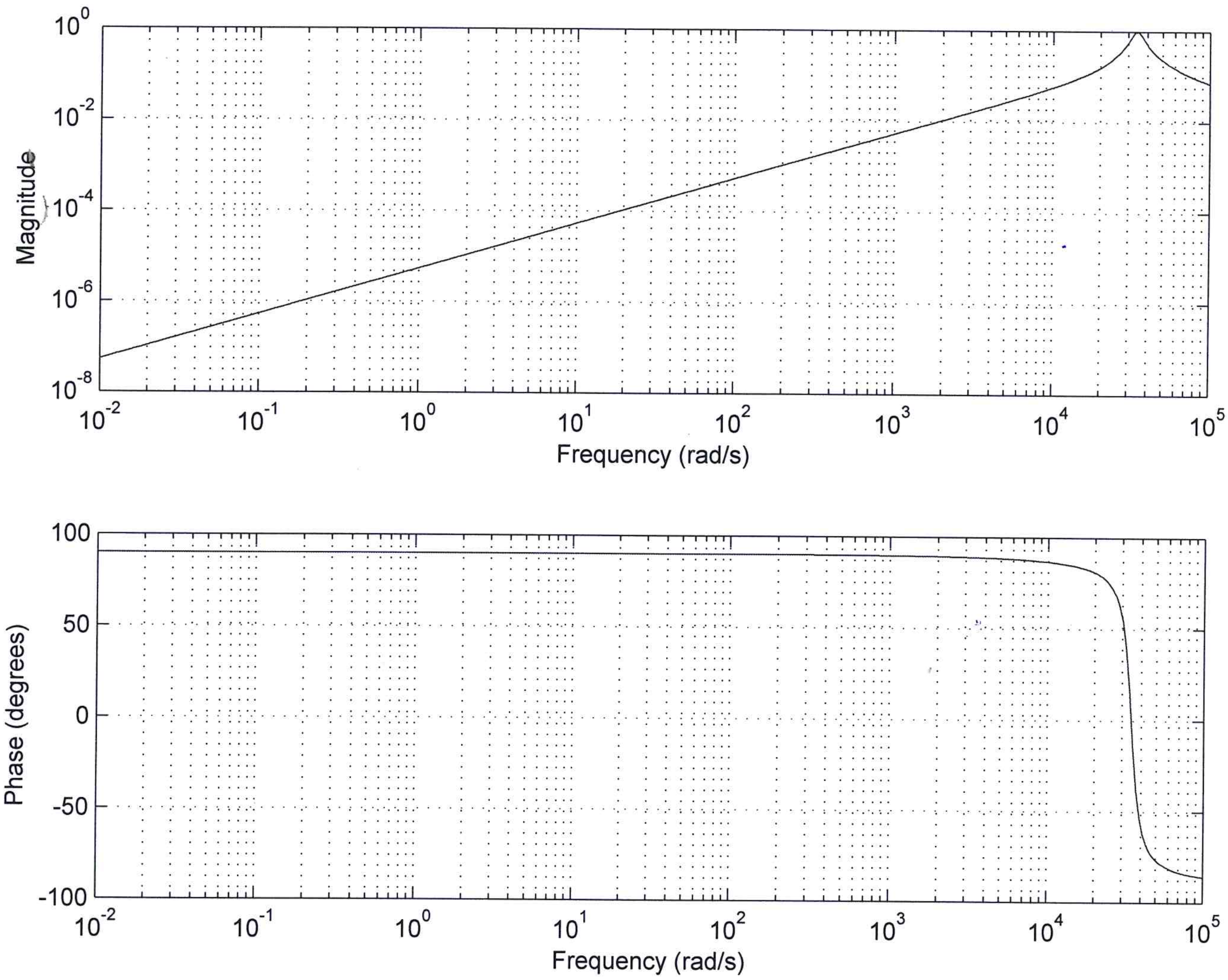

TRASFORMAZIONI $\boxed{\text{PASSA BASSO NUMERICO}} \rightarrow \boxed{\text{FILTRO DESIDERATO NUMERICO}}$ TD1

$H_{LP}(z)$ $H'(z')$

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## 8-5. DIGITAL FREQUENCY TRANSFORMATIONS

In this section, digital frequency transformations will be discussed. These transformations can be used to transform a digital low-pass filter into a digital high-pass, band-pass, or band-stop filter. Hence, once a digital low-pass filter is obtained by using the impulse-invariant method, the other type of filters can then be obtained by using digital frequency transformations. This section is the counterpart of Sec. 8-3 for the digital case.

Consider the transformation

$$z = f(z') \qquad \text{or} \qquad e^{j\omega T} = f(e^{j\omega' T})$$

The function $f(z')$ is required to have the property $|f(e^{j\omega' T})| = 1$ for all $\omega'$. A function with this property will be called a *unit function*. A unit function maps the unit circle on the z'-plane into the unit circle on the z-plane. For illustration, consider the mapping shown in Fig. 8-17(a). It is a two-to-one mapping. We see that both $\omega'_{p1}$ and $-\omega'_{p2}$ on the $\omega'$-axis are mapped into $-\omega_p$ on the $\omega$-axis, both $\omega' = 0$ and $\omega' = \pi/T$ are mapped into $\omega = 0$.

Consider the transfer function $H(z)$ with the amplitude characteristic shown in Fig. 8-17(b). Define

$$H'(z') \triangleq H(z)\Big|_{z=f(z')} = H(f(z'))$$

or

$$H'(e^{j\omega' T}) = H(f(e^{j\omega' T}))$$

The new transfer function $H'(z')$ is obtained from $H(z)$ by the substitution of $z = f(z')$. The amplitude characteristic of $H'(z')$ can be obtained from that of $H(z)$ as shown in Fig. 8-17. For example, the magnitude of $|H'(e^{j\omega' T})|$ at $\omega'_{p1}$ and $-\omega'_{p2}$ is equal to the magnitude of $|H(e^{j\omega T})|$ at $-\omega_p$ that is measured as a. By this process

TD2

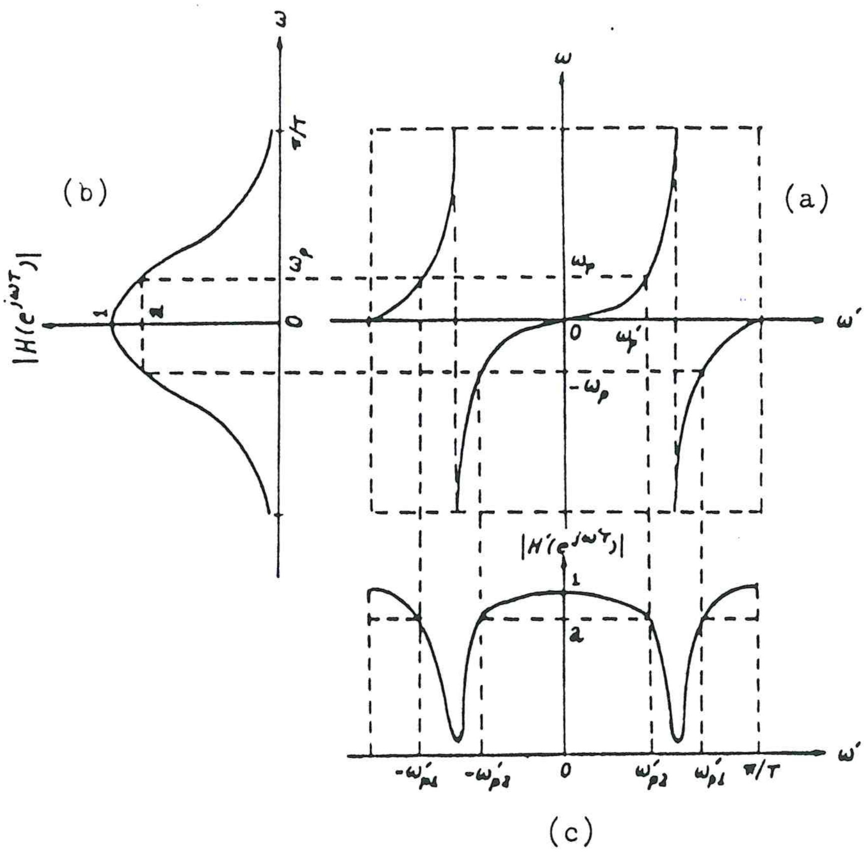


FIG. 8-17. A digital frequency transformation.

the amplitude characteristic of $H'(z')$ can be readily obtained. We see that this transformation converts a low-pass filter into a band-stop filter. This type of transformation is called a *digital frequency transformation.*

Not every function $f(z')$ is a unit function, that is, has the property $|f(e^{j\omega' T})| = 1$ for all $\omega'$. We claim that a unit function must be of the form

$$\pm 1 \qquad \pm z' \qquad \pm z'^{-1} \qquad \pm \frac{z' + b}{bz' + 1} \qquad \pm \frac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1}$$

$$\pm \frac{z'^3 + d_1 z'^2 + d_2 z' + d_3}{d_3 z'^3 + d_2 z'^2 + d_1 z' + 1} \tag{8-46}$$

or $z^n D(-z)/D(z)$, where $n$ is the degree of the polynomial $D(z)$ with

TD3

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real coefficients, and $b$, $c_i$, and $d_i$ are real numbers. We verify this for the last term in (8-46). Indeed, we have

$$f(e^{j\omega'T}) = \frac{e^{3j\omega'T} + d_1 e^{2j\omega'T} + d_2 e^{j\omega'T} + d_3}{d_3 e^{3j\omega'T} + d_2 e^{2j\omega'T} + d_1 e^{j\omega'T} + 1}$$

$$= \frac{e^{3j\omega'T} + d_1 e^{2j\omega'T} + d_2 e^{j\omega'T} + d_3}{e^{3j\omega T}\left(e^{-3j\omega'T} + d_1 e^{-2j\omega'T} + d_2 e^{-j\omega'T} + d_3\right)}$$

$$= \frac{(\cos 3\omega'T + d_1 \cos 2\omega'T + d_2 \cos \omega'T + d_3) + j(\sin 3\omega'T + d_1 \sin 2\omega'T + d_2 \sin \omega'T)}{e^{3j\omega'T}[\cos 3\omega'T + d_1 \cos 2\omega'T + d_2 \cos \omega'T + d_3) - j(\sin 3\omega'T + d_1 \sin 2\omega'T + d_2 \sin \omega'T)]}$$

Since the magnitudes of the real and imaginary parts of the denominator and numerator are the same, we conclude that $|f(e^{j\omega'T})| = 1$ for all $\omega'$. This shows that the functions in (8-46) are indeed unit functions. (See Problem 8-24.)

With this background, we are ready to discuss the required transformation in the design of various digital filters.

## Low-Pass-to-Low-Pass Transformation

Consider the two low-pass filters shown in Fig. 8-18(a); one has a bandwidth $\omega_p$, the other $\omega'_p$. The problem is to find a unit function $z = f(z')$ to achieve this change of bandwidth.

Consider the transformation

$$z = \frac{z' + b}{bz' + 1} = f_\ell(z') \tag{8-47}$$

or

$$e^{j\omega T} = \frac{e^{j\omega'T} + b}{be^{j\omega'T} + 1}$$

It is easy to verify that $f_\ell$ maps $\omega' = 0$ into $\omega = 0$, and $\omega' = \pi/T$ into $\omega = \pi/T$. Now if $b$ is chosen so that $f_\ell$ maps $\omega'_p$ into $\omega_p$, then

TD4

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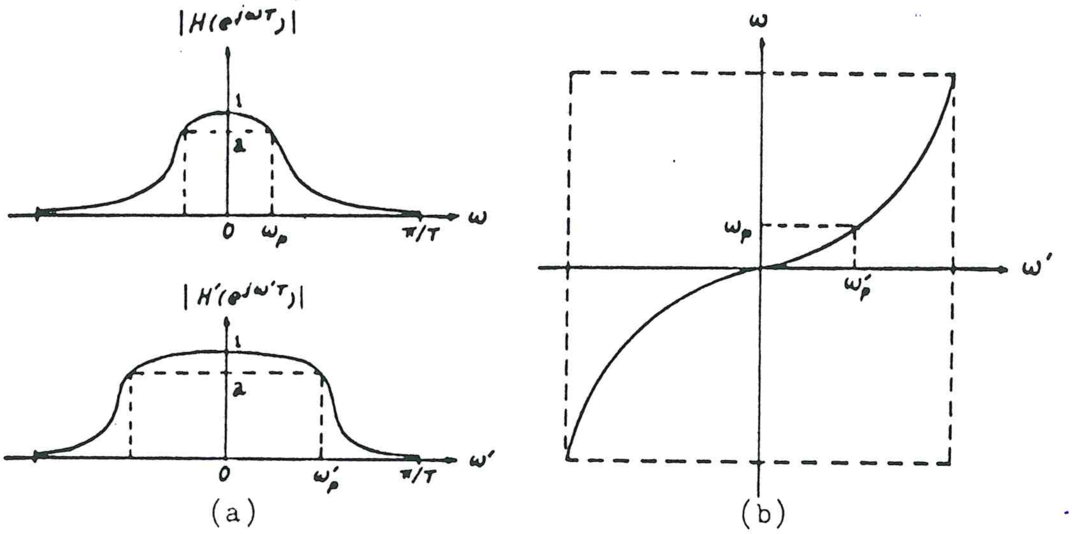


FIG. 8-18. A low-pass-to-low-pass digital frequency transformation.

$f_\ell(z')$ is the desired transformation. We equate

$$e^{j\omega_p T} = \frac{e^{j\omega'_p T} + b}{b e^{j\omega'_p T} + 1}$$

that implies

$$\begin{aligned} b &= \frac{e^{j\omega'_p T} - e^{j\omega_p T}}{e^{j(\omega_p+\omega'_p)T} - 1} \\ &= \frac{e^{j(\omega_p+\omega'_p)T/2}\left[e^{j(\omega'_p-\omega_p)T/2} - e^{-j(\omega'_p-\omega_p)T/2}\right]}{e^{j(\omega_p+\omega'_p)T/2}\left[e^{j(\omega'_p+\omega_p)T/2} - e^{-j(\omega'_p+\omega_p)T/2}\right]} \\ &= \frac{\sin\,(\omega'_p T - \omega_p T)/2}{\sin\,(\omega'_p T + \omega_p T)/2} \end{aligned} \tag{8-48}$$

Hence, using (8-48), the required transformation can be obtained. The substitution of z by $f_\ell(z')$ will convert a low-pass filter of bandwidth $\omega_p$ into a different low-pass filter of bandwidth $\omega'_p$. We plot in Fig. 8-18(b) the transformation as a function of $\omega'$.

TD5



## Low-Pass-to-High-Pass Filter

We discuss now the transformation that transforms a low-pass filter into a high-pass filter. Consider the unit function

$$z = f_h(z') = -\frac{z' + b}{bz' + 1} \tag{8-49}$$

or

$$e^{j\omega T} = -\frac{e^{j\omega' T} + b}{be^{j\omega' T} + 1} \tag{8-50}$$

This function transforms $\omega' = 0$ into $\omega = \pm\pi/T$, and $\omega' = \pm\pi/T$ into $\omega = 0$ as shown in Fig. 8-19(a). If $\omega' = \omega'_p$ is transformed into $\omega = \omega_p$, then we have

$$e^{j\omega_p T} = -\frac{e^{j\omega'_p T} + b}{be^{j\omega'_p T} + 1} \tag{8-51}$$

Simple manipulation yields

$$b = -\frac{\cos(\omega'_p T - \omega_p T)/2}{\cos(\omega'_p T + \omega_p T)/2} \tag{8-52}$$

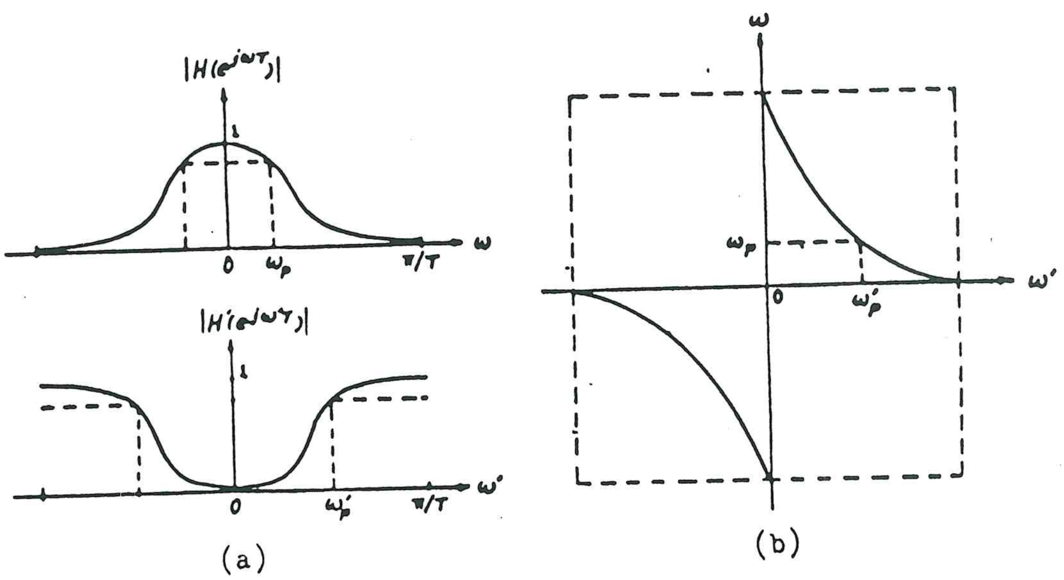


FIG. 8-19. A low-pass-to-high-pass digital frequency transformation.

TD6

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A plot of Eq. (8-50) with b given by (8-52) is shown in Fig. 8-19(b). We note that the b in (8-52) is obtained by transforming $\omega' = \omega'_p$ into $\omega = \omega_p$. If we transform $\omega' = \omega'_p$ into $\omega = -\omega_p$, then we obtain

$$e^{-j\omega_p T} = -\frac{e^{j\omega'_p T} + b}{be^{j\omega'_p T} + 1}$$

which implies

$$b = -\frac{\cos\,(\omega'_p T + \omega_p T)/2}{\cos\,(\omega'_p T - \omega_p T)/2} \tag{8-53}$$

The plot of (8-50) as a function of $\omega'$ with b given by (8-53) will be in the second and fourth quadrants rather than in the first and third quadrants as for the b given by (8-52). The transformation using either (8-52) or (8-53) will give the same amplitude characteristic but different phase characteristics. Hence, the phase characteristics should be the factor to determine which transform to use.

Low-Pass-to-Band-Pass Transformation

The function required in a low-pass-to-band-pass transformation must transform $\omega'_{p1}$ into $\omega_p$ and $\omega'_{p2}$ into $-\omega_p$ as shown in Fig. 8-20(a); therefore, the function must have two parameters or more. Consider

$$z = f_b(z') = -\frac{z'^2 + c_1 z' + c_2}{c_z z'^2 + c_1 z' + 1} \tag{8-54}$$

or

$$e^{j\omega T} = -\frac{e^{2j\omega' T} + c_1 e^{j\omega' T} + c_2}{c_2 e^{2j\omega T} + c_1 e^{j\omega T} + 1} \tag{8-55}$$

This function transforms $\omega' = \pi/T$ into $\omega = \pi/T$ and $\omega' = 0$ into $\omega = -\pi/T$. If the function transforms $\omega'_{p1}$ into $\omega_p$ and $\omega'_{p2}$ into $-\omega_p$, then we have

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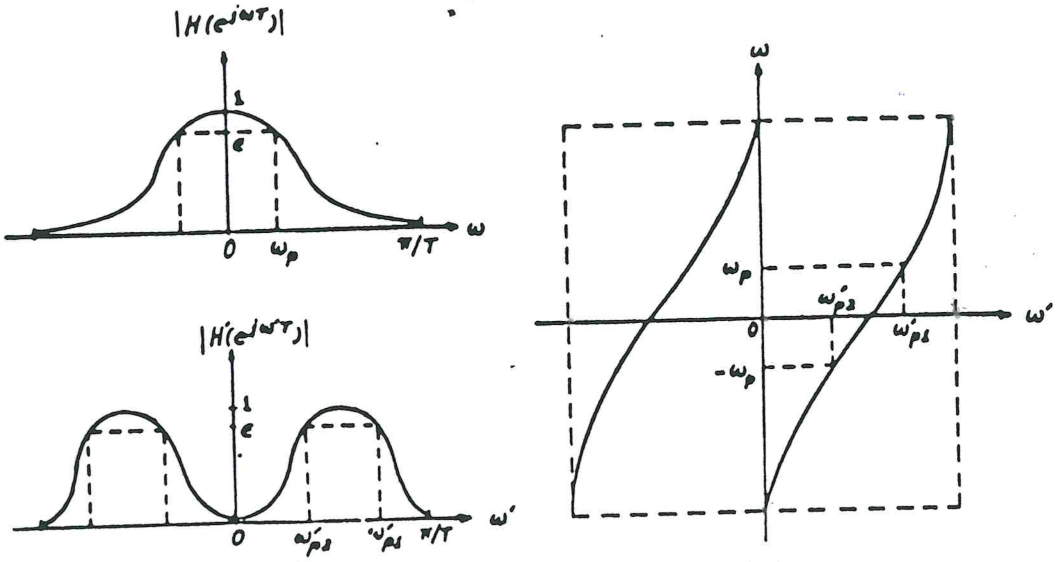


(a) (b)

FIG. 8-20. A low-pass-to-band-pass digital frequency transformation.

$$e^{j\omega_p T} = -\frac{e^{2j\omega'_{p1}T} + c_1 e^{j\omega'_{p1}T} + c_2}{c_2 e^{2j\omega'_{p1}T} + c_1 e^{j\omega'_{p1}T} + 1}$$

$$e^{-j\omega_p T} = -\frac{e^{2j\omega'_{p2}T} + c_1 e^{j\omega'_{p2}T} + c_2}{c_2 e^{2j\omega'_{p2}T} + c_1 e^{j\omega'_{p2}T} + 1}$$

These equations can be arranged as

$$c_1 e^{j\omega'_{p1}T}\left(e^{j\omega_p T} + 1\right) + c_2\left[e^{j(\omega_p + 2\omega'_{p1})T} + 1\right] = -e^{2j\omega'_{p1}T} - e^{j\omega_p T}$$

$$c_1 e^{j(\omega'_{p2} - \omega_p)T}\left(1 + e^{j\omega_p T}\right) + c_2\left[e^{j(-\omega_p + 2\omega'_{p2})T} + 1\right] = -e^{2j\omega'_{p2}T} - e^{-j\omega_p T}$$

The elimination of $c_1$ from these equations and simple manipulation yield

$$c_2 = -\frac{e^{j(\omega'_{p1} - \omega_p)T} - e^{j\omega'_{p2}T}}{e^{j\omega'_{p1}T} - e^{j(\omega'_{p2} - \omega_p)T}}$$

From "One Dimensional Digital Signal ... Chen Marcel Dekker 1979

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TABLE 8-3. Digital Frequency Transformations

| | | |
|---|---|---|
| Low-pass to low-pass | $z = \dfrac{z' + b}{bz' + 1}$ | $b = \dfrac{\sin(\omega'_p T - \omega_p T)/2}{\sin(\omega'_p T + \omega_p T)/2}$ |
| Low-pass to high-pass | $z = \dfrac{-(z' + b)}{bz' + 1}$ | $b = -\dfrac{\cos(\omega'_p T - \omega_p T)/2}{\cos(\omega'_p T + \omega_p T)/2}$ |
| Low-pass to band-pass | $z = -\dfrac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1}$ | $h = \dfrac{\cos(\omega'_{p1} + \omega'_{p2})T/2}{\cos(\omega'_{p1} - \omega'_{p2})T/2}$ <br> $k = \cot(\omega'_{p1} - \omega'_{p2})T/2 \tan \omega_p T/2$ <br> $c_1 = 2h \dfrac{k}{k+1}, \quad c_2 = \dfrac{k-1}{k+1}$ |
| Low-pass to band-stop | $z = \dfrac{z'^2 + c_1 z' + c_2}{c_2 z'^2 + c_1 z' + 1}$ | $h = \dfrac{\cos(\omega'_{q1} + \omega'_{q2})T/2}{\cos(\omega'_{q1} - \omega'_{q2})T/2}$ <br> $k = \tan(\omega'_{q1} - \omega'_{q2})T/2 \tan \omega_q T/2$ <br> $c_1 = -2h \dfrac{1}{k+1}, \quad c_2 = \dfrac{1-k}{1+k}$ |

Let us define

$$k = \frac{(e^{j\omega'_{p1}T} + e^{j\omega'_{p2}T})(1 - e^{-j\omega_p T})}{(e^{j\omega'_{p1}T} - e^{j\omega'_{p2}T})(1 + e^{-j\omega_p T})}$$

$$= \cot \frac{(\omega'_{p1} - \omega'_{p2})T}{2} \tan \frac{\omega_p T}{2} \tag{8-56}$$

Then it can be readily verified that $c_2$ can be expressed as

$$c_2 = \frac{k-1}{k+1} \tag{8-57}$$

Similarly, the constant $c_1$ can be solved as

$$c_1 = -2h \frac{k}{k+1} \tag{8-58}$$

where

$$h = \frac{\cos(\omega'_{p1} + \omega'_{p2})T/2}{\cos(\omega'_{p1} - \omega'_{p2})T/2} \tag{8-59}$$

Using Eqs. (8-56) through (8-59), the parameters $c_1$ and $c_2$ in (8-54) can be computed. The plot of (8-54) as a function of $\omega'$ is shown in Fig. 8-20(b). We see that it is a two-to-one mapping. Note that the point $(\omega'_{p1} + \omega'_{p2})/2$, the midpoint of $\omega'_{p1}$ and $\omega'_{p2}$, is not necessarily mapped into $\omega = 0$.

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da PROAKIS MANOLAKIS, "Digital Signal Processing"



**TABLE 8.13** Frequency Transformation for Digital Filters (Prototype Lowpass Filter Has Cutoff Frequency $\omega_p$)

| Type of Transformation | Transformation | Parameters |
|---|---|---|
| Lowpass | $z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$ | $\omega'_p$ = cutoff frequency of new filter<br>$a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$ |
| Highpass | $z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$ | $\omega'_p$ = cutoff frequency new filter<br>$a = -\dfrac{\cos[(\omega_p - \omega'_p)/2]}{\cos[(\omega_p + \omega'_p)/2]}$ |
| Bandpass | $z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$ | $\omega_l$ = lower cutoff frequency<br>$\omega_u$ = upper cutoff frequency<br>$a_1 = -2\alpha K/(K + 1)$<br>$a_2 = (K - 1)/(K + 1)$ ?<br>$\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$<br>$K = \cot \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_c}{2}$ |
| Bandstop | $z^{-1} \longrightarrow \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-1} - a_1 z^{-1} + 1}$ | $\omega_l$ = lower cutoff frequency<br>$\omega_u$ = upper cutoff frequency<br>$a_1 = -2\alpha/(K + 1)$<br>$a_2 = (1 - K)/(1 + K)$<br>$\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$<br>$K = \tan \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_c}{2}$ |