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Appendix A

Laplace and z-Transforms

The first part of this table contains operational transform pairs, such as the transforms of the derivative of $f(t)$. The second contains a list of explicit transforms. An important point, discussed in Chapter 7, is that $f(t)$ is always zero in this table for $t < 0$. Thus, if the shifting theorems are applied, $f(t)$ no longer "starts" at $t = 0$. For example, if $\tilde{F}(z) = Tz/(z - 1)^2$ is multiplied by z^{-n}, then the modified version of $f(t)$ is a ramp function that is zero until $t = nT$ and equal to $(t - nT)$ thereafter.

Line	Laplace Transform	$f(t)$	z-Transform
A	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$	$f(t)$ $\quad f_m = f(mT)$	$\tilde{F}(z) = \sum_{m=0}^{\infty} f_m z^{-m}$
B	$AF(s)$	$Af(t)$	$A\tilde{F}(z)$
C	$F(s) + G(s)$	$f(t) + g(t)$	$\tilde{F}(z) + \tilde{G}(z)$
D	$sF(s) - f(0+)$	$\frac{d}{dt}f(t)$	—
E	$\frac{F(s)}{s}$	$\int_0^t f(\tau)\,d\tau$	—
F	$-\frac{d}{ds}F(s)$	$tf(t)$	$-Tz\frac{d}{dz}[\tilde{F}(z)]$
G	$F(s + a)$	$e^{-at}f(t);\ a > 0$	$\tilde{F}(ze^{aT})$
H	$e^{-nsT}F(s)$	$f(t - nT);\ n > 0$	$z^{-n}\tilde{F}(z)$
I	$aF(as)$	$f\left(\frac{t}{a}\right);\ a > 0$	$\tilde{F}(z)$ with $\frac{T}{a} \rightarrow T$

Line	$F(s)$	$f(t)$ for $t \geq 0$	$\tilde{F}(z)$
100	1	$\delta(t);\ f_m = \frac{1}{T}$ at $m = 0$	$\frac{1}{T}$
101	$\frac{1}{s}$	$u(t);\ f_m = 1$ for $m \geq 0$	$\frac{z}{z-1}$
102	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
103	$\frac{1}{s^3}$	$\frac{1}{2!}t^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
104	$\frac{1}{s^4}$	$\frac{1}{3!}t^3$	$\frac{T^3 z(z^2+4z+1)}{6(z-1)^4}$
105	$\frac{1}{s^{k+1}}$	$\frac{1}{k!}t^k$	$\lim_{a\to 0} \frac{(-1)^k}{k!}\frac{\partial^k}{\partial a^k}\left(\frac{z}{z-e^{-aT}}\right)$
150	$\frac{1}{s - (1/T)\ln a}$	$a^{t/T}$	$\frac{z}{z-a}$
151	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
152	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
153	$\frac{1}{(s+a)^3}$	$\frac{t^2}{2}e^{-at}$	$\frac{T^2e^{-aT}z}{2(z-e^{-aT})^2} + \frac{T^2e^{-2aT}z}{(z-e^{-aT})^3}$
154	$\frac{1}{(s+a)^{k+1}}$	$\frac{t^k}{k!}e^{-at}$	$\frac{(-1)^k}{k!}\frac{\partial^k}{\partial a^k}\left(\frac{z}{z-e^{-aT}}\right)$
155	$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
156	$\frac{a}{s^2(s+a)}$	$t - \frac{1-e^{-at}}{a}$	$\frac{Tz}{(z-1)^2} - \frac{(1-e^{-aT})z}{a(z-1)(z-e^{-aT})}$
157	$\frac{a}{s^3(s+a)}$	$\frac{1}{2!}\left(t^2 - \frac{2}{a}t + \frac{2}{a^2} - \frac{2}{a^2}e^{-at}\right)$	$\frac{T^2z}{(z-1)^3} + \frac{(aT-2)Tz}{2a(z-1)^2} + \frac{z}{a^2(z-1)} - \frac{z}{a^2(z-e^{-aT})}$

Line	$F(s)$	$f(t)$ for $t \geq 0$	$\tilde{F}(z)$
158	$\dfrac{a}{s^{k+1}(s+a)}$	$\dfrac{1}{k!}\left[t^k - \dfrac{k}{a}t^{k-1} + \dfrac{k(k-1)}{a^2}t^{k-2} - \cdots + (-1)^{k-1}\dfrac{k!}{a^k}t + (-1)^k\dfrac{k!}{a^k}\right] + (-1)^{k+1}\dfrac{e^{-at}}{a^k}$	$\dfrac{(-1)^{k+1}}{a^k}\dfrac{1}{1-e^{-aT}z^{-1}} + \dfrac{a}{k!}\lim_{x\to 0}\dfrac{\partial^k}{\partial x^k}\left[\dfrac{1}{(x+a)(1-e^{Tx}z^{-1})}\right]$
200	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$\dfrac{z}{z-e^{-aT}} - \dfrac{z}{z-e^{-bT}}$
201	$\dfrac{(b-a)(s+c)}{(s+a)(s+b)}$	$(c-a)e^{-at} + (b-c)e^{-bt}$	$\dfrac{(c-a)z}{z-e^{-aT}} + \dfrac{(b-c)z}{z-e^{-bT}}$
202	$\dfrac{\beta}{s^2+\beta^2}$	$\sin \beta t$	$\dfrac{z \sin \beta T}{z^2 - 2z\cos \beta T + 1}$
203	$\dfrac{s}{s^2+\beta^2}$	$\cos \beta t$	$\dfrac{z(z - \cos \beta T)}{z^2 - 2z\cos \beta T + 1}$
204	$\dfrac{\beta}{s^2-\beta^2}$	$\sinh \beta t$	$\dfrac{z \sinh \beta T}{z^2 - 2z\cosh \beta T + 1}$
205	$\dfrac{s}{s^2-\beta^2}$	$\cosh \beta t$	$\dfrac{z(z - \cosh \beta T)}{z^2 - 2z\cosh \beta T + 1}$
206	$\dfrac{\beta}{(s+a)^2+\beta^2}$	$e^{-at}\sin \beta t$	$\dfrac{ze^{-aT}\sin \beta T}{z^2 - 2ze^{-aT}\cos \beta T + e^{-2aT}}$
207	$\dfrac{s+a}{(s+a)^2+\beta^2}$	$e^{-at}\cos \beta t$	$\dfrac{z^2 - ze^{-aT}\cos \beta T}{z^2 - 2ze^{-aT}\cos \beta T + e^{-2aT}}$
208	$\dfrac{\beta s}{(s+a)^2+\beta^2}$	$e^{-at}(\beta \cos \beta t - a \sin \beta t)$	$\dfrac{\beta z^2 - ze^{-aT}(\beta \cos \beta T + a \sin \beta T)}{z^2 - 2ze^{-aT}\cos \beta T + e^{-2aT}}$
300	$\dfrac{ab}{s(s+a)(s+b)}$	$1 + \dfrac{b}{a-b}e^{-at} - \dfrac{a}{a-b}e^{-bt}$	$\dfrac{z}{z-1} + \dfrac{bz}{(a-b)(z-e^{-aT})} - \dfrac{az}{(a-b)(z-e^{-bT})}$

Line	$F(s)$	$f(t)$ for $t \geq 0$	$\tilde{F}(z)$
301	$\dfrac{ab(s+c)}{s(s+a)(s+b)}$	$c + \dfrac{b(c-a)}{a-b}e^{-at} + \dfrac{a(b-c)}{a-b}e^{-bt}$	$\dfrac{cz}{z-1} + \dfrac{b(c-a)z}{(a-b)(z-e^{-aT})} + \dfrac{a(b-c)z}{(a-b)(z-e^{-bT})}$
302	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)} + \dfrac{e^{-bt}}{(a-b)(c-b)} + \dfrac{e^{-ct}}{(a-c)(b-c)}$	$\dfrac{z}{(b-a)(c-a)(z-e^{-aT})} + \dfrac{z}{(a-b)(c-b)(z-e^{-bT})} + \dfrac{z}{(a-c)(b-c)(z-e^{-cT})}$
303	$\dfrac{s+d}{(s+a)(s+b)(s+c)}$	$\dfrac{(d-a)}{(b-a)(c-a)}e^{-at} + \dfrac{(d-b)}{(a-b)(c-b)}e^{-bt} + \dfrac{(d-c)}{(a-c)(b-c)}e^{-ct}$	$\dfrac{(d-a)z}{(b-a)(c-a)(z-e^{-aT})} + \dfrac{(d-b)z}{(a-b)(c-b)(z-e^{-bT})} + \dfrac{(d-c)z}{(a-c)(b-c)(z-e^{-cT})}$
304	$\dfrac{a^2}{s(s+a)^2}$	$1 - (1+at)e^{-at}$	$\dfrac{z}{z-1} - \dfrac{z}{z-e^{-aT}} - \dfrac{aTe^{-aT}z}{(z-e^{-aT})^2}$
305	$\dfrac{a^2(s+b)}{s(s+a)^2}$	$b - be^{-at} + a(a-b)te^{-at}$	$\dfrac{bz}{z-1} - \dfrac{bz}{z-e^{-aT}} + \dfrac{a(a-b)Te^{-aT}z}{(z-e^{-aT})^2}$
306	$\dfrac{(a-b)^2}{(s+b)(s+a)^2}$	$e^{-bt} - e^{-at} + (b-a)te^{-at}$	$\dfrac{z}{z-e^{-bT}} - \dfrac{z}{z-e^{-aT}} + \dfrac{(a-b)Te^{-aT}z}{(z-e^{-aT})^2}$
307	$\dfrac{(a-b)^2(s+c)}{(s+b)(s+a)^2}$	$(c-b)e^{-bt} + (b-c)e^{-at} - (a-b)(c-a)te^{-at}$	$\dfrac{(c-b)z}{z-e^{-bT}} + \dfrac{(b-c)z}{z-e^{-aT}} - \dfrac{(a-b)(c-a)Te^{-aT}z}{(z-e^{-aT})^2}$

Line	$F(s)$	$f(t)$ for $t \ge 0$	$\tilde{F}(z)$
308	$\dfrac{\beta^2}{s(s^2 - \beta^2)}$	$\cosh \beta t - 1$	$\dfrac{z(z - \cosh \beta T)}{z^2 - 2z \cosh \beta T + 1} - \dfrac{z}{z - 1}$
309	$\dfrac{\beta^2}{s(s^2 + \beta^2)}$	$1 - \cos \beta t$	$\dfrac{z}{z - 1} - \dfrac{z(z - \cos \beta T)}{z^2 - 2z \cos \beta T + 1}$
310	$\dfrac{\beta^2(s + a)}{s(s^2 + \beta^2)}$	$a - a \sec \theta \cos(\beta t + \theta)$ where $\theta = \tan^{-1}\left(\dfrac{\beta}{a}\right)$	$\dfrac{az}{z - 1} - \dfrac{az^2 - az \sec \theta \cos(\beta T - \theta)}{z^2 - 2z \cos \beta T + 1}$
311	$\dfrac{a^2 + \beta^2}{s[(s + a)^2 + \beta^2]}$	$1 - e^{-at} \sec \theta \cos(\beta t + \theta)$ where $\theta = \tan^{-1}\left(-\dfrac{a}{\beta}\right)$	$\dfrac{z}{z - 1} - \dfrac{z^2 - ze^{-aT} \sec \theta \cos(\beta T - \theta)}{z^2 - 2ze^{-aT} \cos \beta T + e^{-2aT}}$
312	$\dfrac{(a^2 + \beta^2)(s + b)}{s[(s + a)^2 + \beta^2]}$	$b - be^{-at} \sec \theta \cos(\beta t + \theta)$ where $\theta = \tan^{-1}\left(\dfrac{a^2 + \beta^2 - ab}{b\beta}\right)$	$\dfrac{bz}{z - 1} - \dfrac{b[z^2 - ze^{-aT} \sec \theta \cos(\beta T - \theta)]}{z^2 - 2ze^{-aT} \cos \beta T + e^{-2aT}}$
313	$\dfrac{(a - b)^2 + \beta^2}{(s + b)[(s + a)^2 + \beta^2]}$	$e^{-bt} - e^{-at} \sec \theta \cos(\beta t + \theta)$ where $\theta = \tan^{-1}\left(\dfrac{b - a}{\beta}\right)$	$\dfrac{z}{z - e^{-bT}} - \dfrac{z^2 - ze^{-aT} \sec \theta \cos(\beta T - \theta)}{z^2 - 2ze^{-aT} \cos \beta T + e^{-2at}}$
314	$\dfrac{[(a - b)^2 + \beta^2](s + \alpha)}{(s + b)[(s + a)^2 + \beta^2]}$	$(\alpha - b)e^{-bt} - (\alpha - b)e^{-at} \sec \theta \cos(\beta t + \theta)$ where $\theta = \tan^{-1}\left(\dfrac{(\alpha - a)(b - a) + \beta^2}{(\alpha - b)\beta}\right)$	$\dfrac{(\alpha - b)z}{z - e^{-bT}} - \dfrac{(\alpha - b)[z^2 - ze^{-aT} \sec \theta \cos(\beta T - \theta)]}{z^2 - 2ze^{-aT} \cos \beta T + e^{-2aT}}$
315	$\dfrac{[(a - b)^2 + \beta^2](s^2 + \alpha s + \gamma)}{(s + b)[(s + a)^2 + \beta^2]}$	$(b^2 - b\alpha + \gamma)e^{-bt} + k^2e^{-at} \sec \theta \cos(\beta t + \theta)$ where $k^2 = a^2 + \beta^2 - 2ab + b\alpha - \gamma$, $\theta = \tan^{-1}\left(\dfrac{ak^2 - (a^2 + \beta^2)(\alpha - b) + \gamma(2a - b)}{\beta k^2}\right)$	$\dfrac{(b^2 - b\alpha + \gamma)z}{z - e^{-bT}} + \dfrac{k^2[z^2 - ze^{-aT} \sec \theta \cos(\beta T - \theta)]}{z^2 - 2ze^{-aT} \cos \beta T + e^{-2aT}}$

Line	$F(s)$	$f(t)$ for $t \ge 0$	$\tilde{F}(z)$
400	$\dfrac{a^3}{s^2(s + a)^2}$	$at - 2 + (at + 2)e^{-at}$	$\dfrac{(aT + 2)z - 2z^2}{(z - 1)^2} + \dfrac{2z}{z - e^{-aT}} + \dfrac{aTe^{-aT}z}{(z - e^{-aT})^2}$
401	$\dfrac{a^2b^2}{s^2(s + a)(s + b)}$	$abt - (a + b) - \dfrac{b^2}{a - b}e^{-at} + \dfrac{a^2}{a - b}e^{-bt}$	$\dfrac{abTz}{(z - 1)^2} - \dfrac{(a + b)z}{z - 1} - \dfrac{b^2z}{(a - b)(z - e^{-aT})} + \dfrac{a^2z}{(a - b)(z - e^{-bT})}$
402	$\dfrac{a^2b^2(s + c)}{s^2(s + a)(s + b)}$	$abct + [ab - c(a + b)] - \dfrac{b^2(c - a)}{a - b}e^{-at} - \dfrac{a^2(b - c)}{a - b}e^{-bt}$	$\dfrac{abcTz}{(z - 1)^2} + \dfrac{ab - c(a + b)z}{z - 1} - \dfrac{b^2(c - a)z}{(a - b)(z - e^{-aT})} - \dfrac{a^2(b - c)z}{(a - b)(z - e^{-bT})}$
403	$\dfrac{abc}{s(s + a)(s + b)(s + c)}$	$1 - \dfrac{bc}{(b - a)(c - a)}e^{-at} - \dfrac{ca}{(c - b)(a - b)}e^{-bt} - \dfrac{ab}{(a - c)(b - c)}e^{-ct}$	$\dfrac{z}{z - 1} - \dfrac{bcz}{(b - a)(c - a)(z - e^{-aT})} - \dfrac{caz}{(c - b)(a - b)(z - e^{-bT})} - \dfrac{abz}{(a - c)(b - c)(z - e^{-cT})}$
404	$\dfrac{abc(s + d)}{s(s + a)(s + b)(s + c)}$	$d - \dfrac{bc(d - a)}{(b - a)(c - a)}e^{-at} - \dfrac{ca(d - b)}{(c - b)(a - b)}e^{-bt} - \dfrac{ab(d - c)}{(a - c)(b - c)}e^{-ct}$	$\dfrac{dz}{z - 1} - \dfrac{bc(d - a)z}{(b - a)(c - a)(z - e^{-aT})} - \dfrac{ca(d - b)z}{(c - b)(a - b)(z - e^{-bT})} - \dfrac{ab(d - c)z}{(a - c)(b - c)(z - e^{-cT})}$
405	$\dfrac{a^2b}{s(s + b)(s + a)^2}$	$1 - \dfrac{a^2}{(a - b)^2}e^{-bt} + \dfrac{ab + b(a - b)}{(a - b)^2}e^{-at} + \dfrac{ab}{a - b}te^{-at}$	$\dfrac{z}{z - 1} - \dfrac{a^2z}{(a - b)^2(z - e^{-bT})} + \dfrac{[ab + b(a - b)]z}{(a - b)^2(z - e^{-aT})} + \dfrac{abTe^{-aT}z}{(a - b)(z - e^{-aT})^2}$
406	$\dfrac{a^2b(s + c)}{s(s + b)(s + a)^2}$	$c + \dfrac{a^2(b - c)}{(a - b)^2}e^{-bt} + \dfrac{ab(c - a) + bc(a - b)}{(a - b)^2}e^{-at} + \dfrac{ab(c - a)}{a - b}te^{-at}$	$\dfrac{cz}{z - 1} + \dfrac{a^2(b - c)z}{(a - b)^2(z - e^{-bT})} + \dfrac{[ab(c - a) + bc(a - b)]z}{(a - b)^2(z - e^{-aT})} + \dfrac{ab(c - a)Te^{-aT}z}{(a - b)(z - e^{-aT})^2}$

Line	$F(s)$	$f(t)$ for $t \geq 0$	$\tilde{F}(z)$
407	$\dfrac{(a^2+\beta^2)^2}{s^2[(s+a)^2+\beta^2]}$	$(a^2+\beta^2)t - 2a + 2ae^{-at}\sec\theta\cos(\beta t+\theta)$ where $\theta = \tan^{-1}\left(\dfrac{\beta^2-a^2}{2a\beta}\right)$	$\dfrac{[(a^2+\beta^2)T+2a]z - 2az^2}{(z-1)^2}$ $+\dfrac{2a[z^2 - ze^{-aT}\sec\theta\cos(\beta T-\theta)]}{z^2-2ze^{-aT}\cos\beta T + e^{-2aT}}$
408	$\dfrac{(a^2+\beta^2)^2(s+b)}{s^2[(s+a)^2+\beta^2]}$	$b(a^2+\beta^2)t + k^2 - k^2e^{-at}\sec\theta\cos(\beta t+\theta)$ where $k^2 = a^2+\beta^2-2ab$ $\theta = \tan^{-1}\left(-\dfrac{ak^2+b(a^2+\beta^2)}{\beta k^2}\right)$	$\dfrac{[bT(a^2+\beta^2)-k^2]z + k^2z^2}{(z-1)^2}$ $-\dfrac{k^2[z^2-ze^{-aT}\sec\theta\cos(\beta T-\theta)]}{z^2-2ze^{-aT}\cos\beta T+e^{-2aT}}$
500	$\dfrac{(abc)^2}{s^2(s+a)(s+b)(s+c)}$	$abct - (bc+ca+ab) + \dfrac{b^2c^2}{(b-a)(c-a)}e^{-at}$ $+\dfrac{c^2a^2}{(c-b)(a-b)}e^{-bt} + \dfrac{a^2b^2}{(a-c)(b-c)}e^{-ct}$	$\dfrac{abcTz}{(z-1)^2} - \dfrac{(bc+ca+ab)z}{z-1} + \dfrac{b^2c^2z}{(b-a)(c-a)(z-e^{-aT})}$ $+\dfrac{c^2a^2z}{(c-b)(a-b)(z-e^{-bT})} + \dfrac{a^2b^2z}{(a-c)(b-c)(z-e^{-cT})}$
501	$\dfrac{(abc)^2(s+d)}{s^2(s+a)(s+b)(s+c)}$	$abcdt + [abc-(bc+ca+ab)d] + \dfrac{b^2c^2(d-a)}{(b-a)(c-a)}e^{-at}$ $+\dfrac{c^2a^2(d-b)}{(c-b)(a-b)}e^{-bt} + \dfrac{a^2b^2(d-c)}{(a-c)(b-c)}e^{-ct}$	$\dfrac{abcdTz}{(z-1)^2} + \dfrac{[abc-(bc+ca+ab)d]z}{z-1}$ $+\dfrac{b^2c^2(d-a)z}{(b-a)(c-a)(z-e^{-aT})} + \dfrac{c^2a^2(d-b)z}{(c-b)(a-b)(z-e^{-bT})}$ $+\dfrac{a^2b^2(d-c)z}{(a-c)(b-c)(z-e^{-cT})}$
502	$\dfrac{(a^2b)^2}{s^2(s+b)(s+a)^2}$	$a^2bt - [ab+a(a+b)] + \dfrac{a^4}{(a-b)^2}e^{-bt}$ $-\dfrac{ab^2(3a-2b)}{(a-b)^2}e^{-at} - \dfrac{a^2b^2}{a-b}te^{-at}$	$\dfrac{a^2bTz}{(z-1)^2} - \dfrac{[ab+a(a+b)]z}{z-1} + \dfrac{a^4z}{(a-b)^2(z-e^{-bT})}$ $-\dfrac{ab^2(3a-2b)z}{(a-b)^2(z-e^{-aT})} - \dfrac{a^2b^2Te^{-aT}z}{(a-b)(z-e^{-aT})^2}$