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SECONDA UNIVERSITA' DEGLI STUDI DI NAPOLI
DIPARTIMENTO DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE
Scuola Politecnica e delle Scienze di Base

Teoria dei Segnali/Telecomunicazioni 2
Ia PROVA INTRACORSO
Prof. Francesco A. N. Palmieri
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(SOLUZIONI)

Schizzare i seguenti segnali e valutarne l'energia e la potenza. Inoltre se ne valuti e se ne schizzi al meglio la trasformata di Fourier.
(1) $s(t) = e^{-|t-1|}$;
(2) $s(t) = 1 - \sin^2(\pi t + \frac{\pi}{4})$;
(3) Per il segnale periodico

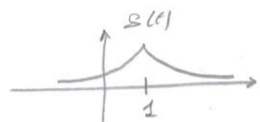
$$s(t) = \sum_{k=-\infty}^{\infty} \left[\Pi\left(\frac{t}{4} - k\right) - \Lambda\left(t - 4k\right) \right], \tag{1}$$

eseguire uno schizzo, valutare la trasformata di Fourier, lo spettro di potenza e la autocorrelazione.
(4) Dati due segnali $x(t) = e^{-t}u(t+1)$ e $y(t) = u(t+2) - u(t-7)$. Valutare la convoluzione lineare usando il metodo grafico.
(5) Si consideri il segnale

$$X(t) = 1 + S(t)\sin^2(5\pi t), \tag{2}$$

dove $S(t)$ è un processo aleatorio SSL avente spettro di potenza piatto nella banda $[f_1\ f_2]$ Hz e potenza P_0. Ricavare la autocorrelazione di $S(t)$, la autocorrelazione e lo spettro di potenza medio di $X(t)$ commentando sulla stazionarietá.

(1) $s(t) = e^{-|t-1|}$



$$E_s = \int_{-\infty}^{+\infty} s(t)^2\,dt = \int_{-\infty}^{1} \left(e^{t-1}\right)^2 dt + \int_{1}^{\infty} \left(e^{-t+1}\right)^2 dt = 2\int_{-\infty}^{1} e^{2(t-1)}\,dt =$$

$$= 2e^{-2}\int_{-\infty}^{1} e^{2t}\,dt = 2e^{-2} \left[\frac{e^{2t}}{2}\right]_{-\infty}^{1} = e^{-2} e^{2} = 1 \quad ; \quad P_s = 0$$

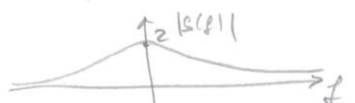
$$S(f) = e^{-j2\pi f}\, \mathcal{F}\left[e^{-|t|}\right] = e^{-j2\pi f}\left[\int_{-\infty}^{0} e^{t} e^{-j2\pi f t}\,dt + \int_{0}^{\infty} e^{-t} e^{-j2\pi f t}\,dt\right]$$

$$= e^{-j2\pi f}\left[\int_{-\infty}^{0} e^{(1-j2\pi f)t}\,dt + \int_{0}^{\infty} e^{-(1+j2\pi f)t}\,dt\right]$$

$$= e^{-j2\pi f}\left[\left[\frac{e^{(1-j2\pi f)t}}{1-j2\pi f}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+j2\pi f)t}}{-(1+j2\pi f)}\right]_{0}^{\infty}\right]$$

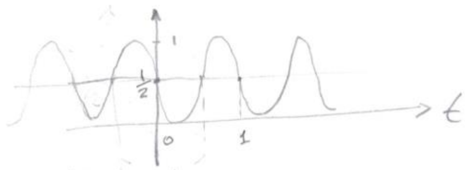
$$= e^{-j2\pi f}\left(\frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f}\right) = \frac{1+j2\pi f + 1 - j2\pi f}{1+4\pi^2 f^2}\, e^{-j2\pi f}$$

$$= 2\,\frac{e^{-j2\pi f}}{1+4\pi^2 f^2} \quad ; \quad |S(f)| = \frac{2}{1+4\pi^2 f^2}, \quad \angle S(f) = -2\pi f$$



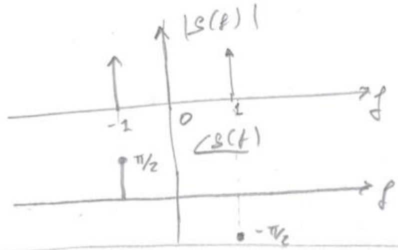
(2) $s(t) = 1 - \sin^2(\pi t + \frac{\pi}{4}) = 1 - \left(1 - \cos^2(\pi t + \frac{\pi}{4})\right) = \cos^2(\pi t + \frac{\pi}{4})$ S2

$= \frac{1}{2} + \frac{1}{2}\cos(2\pi t + \frac{\pi}{2})$



$$S(f) = \frac{1}{2}\delta(f) + \frac{1}{4}e^{-j\frac{\pi}{2}}\delta(f-1) + \frac{1}{4}e^{j\frac{\pi}{2}}\delta(f+1)$$

$$= \frac{1}{2}\delta(f) - \frac{j}{4}\delta(f-1) + \frac{j}{4}\delta(f+1)$$



Segnale di potenza. Poiché periodico periodo $T = 1$

$P_s = \frac{\text{Energia in un periodo}}{\text{periodo}} = \frac{\mathcal{E}_T}{1}$

$$\mathcal{E}_T = \int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos(2\pi t + \frac{\pi}{2})\right)^2 dt = \int_0^1 \left(\frac{1}{4} + \frac{1}{4}\cos^2(2\pi t + \frac{\pi}{2}) + \frac{1}{2}\cos(2\pi t + \frac{\pi}{2})\right) dt$$

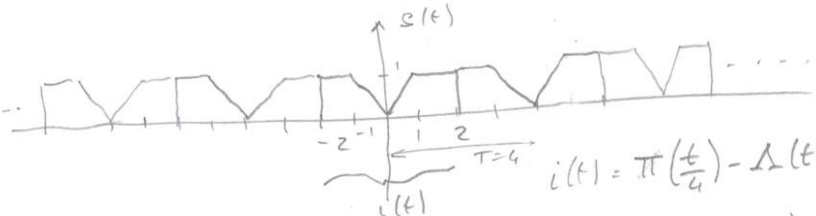
$$= \frac{1}{4} + \frac{1}{8}\int_0^1 dt + \frac{1}{8}\underbrace{\int_0^1 \cos(4\pi t + \pi)\, dt}_{0} + \frac{1}{2}\underbrace{\int_0^1 \cos(2\pi t + \frac{\pi}{2})\, dt}_{0}$$

$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

S3

(3) $s(t) = \sum_{k=-\infty}^{+\infty} \left(\Pi\left(\frac{t}{4} - k\right) - \Lambda(t-4k) \right)$

$= \sum_{k=-\infty}^{+\infty} \left(\Pi\left(\frac{t-4k}{4}\right) - \Lambda(t-4k) \right)$



$i(t) = \Pi\left(\frac{t}{4}\right) - \Lambda(t)$

$= \sum_{k=-\infty}^{+\infty} i(t-4k) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{4} \sum_{n=-\infty}^{+\infty} I\left(\frac{n}{4}\right) \delta\left(f - \frac{n}{4}\right)$

$I(f) = 4\,\mathrm{sinc}\,4f - \mathrm{sinc}^2 f$

$S(f) = \frac{1}{4} \sum_{n=-\infty}^{+\infty} \Big(\underbrace{4\,\mathrm{sinc}\,4\frac{n}{4}}_{\substack{=0 \ \forall n \neq 0 \\ =1 \ n=0}} - \mathrm{sinc}^2 \frac{n}{4} \Big) \delta\left(f - \frac{n}{4}\right)$

$= \frac{1}{4} \sum_{n=-\infty}^{+\infty} 4\delta[n]\,\delta\left(f - \frac{n}{4}\right) - \frac{1}{4} \sum_{n=-\infty}^{+\infty} \mathrm{sinc}^2 \frac{n}{4}\, \delta\left(f - \frac{n}{4}\right)$

$= \delta(f) - \frac{1}{4} \sum_{n=-\infty}^{+\infty} \mathrm{sinc}^2 \frac{n}{4}\, \delta\left(f - \frac{n}{4}\right)$

Lo spettro di potenza di un segnale periodico sappiamo che è $P(f) = \frac{1}{T^2} \sum_{k=-\infty}^{+\infty} \left| I\left(\frac{k}{T}\right)\right|^2 \delta\left(f - \frac{k}{T}\right)$

$\left| I\left(\frac{k}{4}\right)\right|^2 = \left(4\,\mathrm{sinc}\,4\frac{k}{4} - \mathrm{sinc}^2\frac{k}{4}\right)^2 = \left(4\delta[k] - \mathrm{sinc}^2 \frac{k}{4}\right)^2$

$= 16\delta[k] + \mathrm{sinc}^4\frac{k}{4} - 8\delta[k]\,\mathrm{sinc}^2\frac{k}{4} =$

$= 16\delta[k] + \mathrm{sinc}^4\frac{k}{4} - 8\delta[k] = 8\delta[k] + \mathrm{sinc}^4\frac{k}{4}$

$P_s(f) = \frac{1}{16} \sum_{k=-\infty}^{+\infty} \left(8\delta[k] + \mathrm{sinc}^4\frac{k}{4}\right) \delta\left(f - \frac{k}{4}\right)$

$= \frac{1}{2}\delta(f) + \frac{1}{16} \sum_{k=-\infty}^{+\infty} \mathrm{sinc}^4 \frac{k}{4}\, \delta\left(f - \frac{k}{4}\right)$

Lo stesso risultato si poteva ottenere sfruttando $p(t)=\sum_{k=-\infty}^{+\infty} \Lambda(t-4k)$ 84

$$s(t) = 1 - p(t)$$

$$r_s(\tau) \triangleq \lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} s(t)s(t-\tau)\,dt = \lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} (1-p(t))(1-p(t-\tau))\,dt$$

$$= \lim_{\alpha\to} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} (1 - p(t) - p(t-\tau) + p(t)p(t-\tau))\,dt$$

$$= 1 - \underbrace{\lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} p(t)\,dt}_{\substack{\text{media} \\ \langle p(t)\rangle}} - \underbrace{\lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} p(t-\tau)\,dt}_{\langle p(t)\rangle} + \underbrace{\lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} p(t)p(t-\tau)\,dt}_{r_p(\tau)}$$

$$= 1 - 2\langle p(t)\rangle + r_p(\tau) \qquad \langle p(t)\rangle = \frac{1}{4}$$

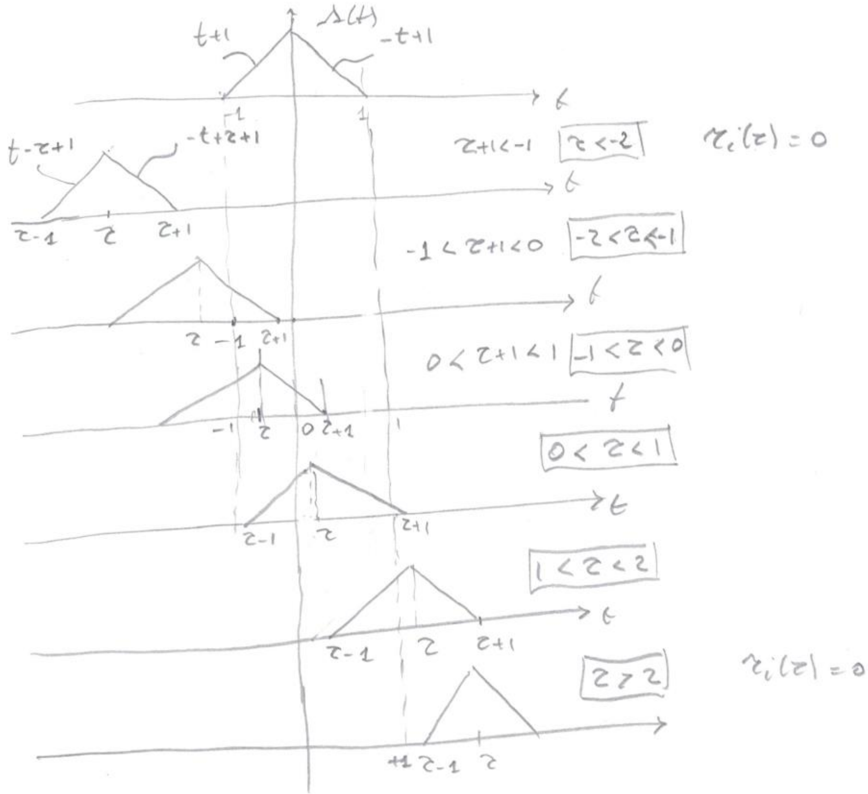
$$= 1 - \frac{1}{2} + r_p(\tau) = \frac{1}{2} + r_p(\tau)$$

$$P_s(f) = \frac{1}{2}\delta(f) + P_p(f) \quad ; \quad P_p(f) = \frac{1}{16}\sum_{k=-\infty}^{+\infty} \operatorname{sinc}^4\frac{k}{4}\,\delta\!\left(f-\frac{k}{4}\right)$$

Per la autocorrelazione abbiamo

$$r_p(\tau) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} r_i(\tau - kT) \qquad r_i(\tau) = \int_{-\infty}^{+\infty} \Lambda(t)\Lambda(t-\tau)\,dt$$

calcolo di $r_i(\tau)=\int_{-\infty}^{+\infty}\Delta(t)\,\Delta(t-\tau)\,dt$



$-2<\tau<-1$

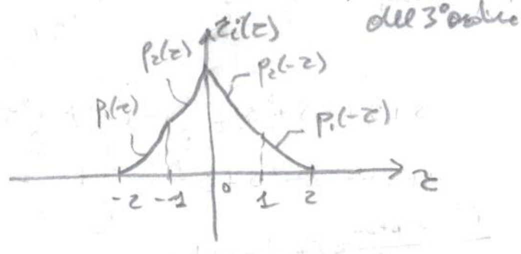
$$r_i(\tau)=\int_{-1}^{\tau+1}(t+1)(-t+\tau+1)\,dt = p_1(\tau)$$ polinomio del 3° ordine

$-1<\tau<0$

$$r_i(\tau)=\int_{-1}^{\tau}(t+1)(t-\tau+1)\,dt+\int_{\tau}^{0}(t+1)(-t+\tau+1)\,dt+\int_{0}^{\tau+1}(-t+1)(-t+\tau+1)\,dt = p_2(\tau)$$ polinomio del 3° ordine

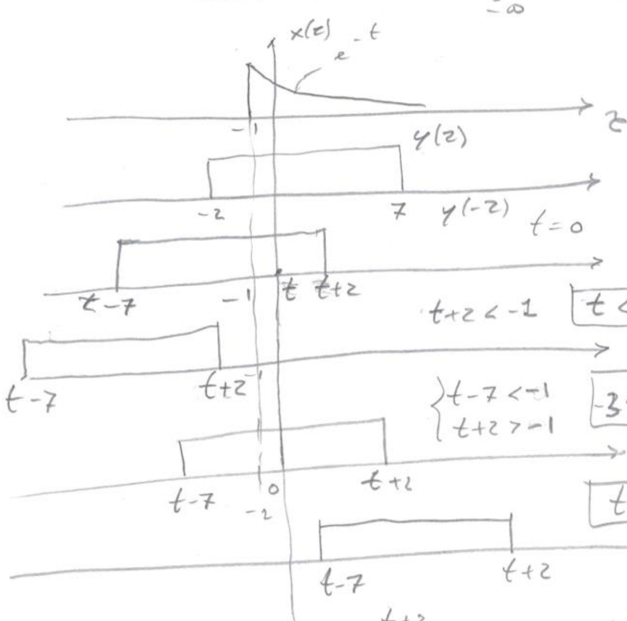
$0<\tau<1$ $\quad r_i(\tau)=p_2(-\tau)$

$1<\tau<2$ $\quad r_i(\tau)=p_1(-\tau)$



(4) $x(t) = e^{-t}u(t+1)$; $y(t) = u(t+2) - u(t-7)$

$$z(t) = (x*y)(t) = \int_{-\infty}^{+\infty} x(\tau)\,y(t-\tau)\,d\tau$$



$t+2 < -1$ $\boxed{t < -3}$ $z(t) = 0$

$\begin{cases} t-7 < -1 \\ t+2 > -1 \end{cases}$ $\boxed{-3 < t < 6}$

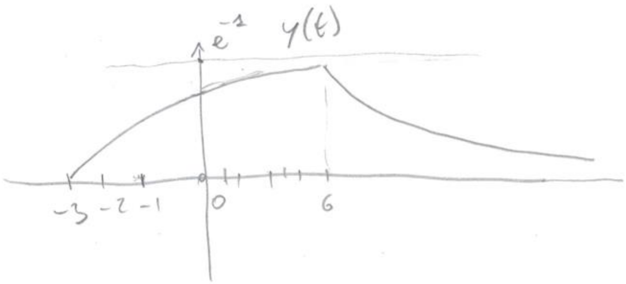
$\boxed{t > 7}$

$\boxed{-2 < t < 7}$ $z(t) = \int_{-1}^{t+2} e^{-t}\,dt = \frac{e^{-t}}{-1}\Big]_{-1}^{t+2} = e^{-1} - e^{-t-2}$

$\boxed{t > 7}$ $z(t) = \int_{t-7}^{t+2} e^{-t}\,dt = \frac{e^{-t}}{-1}\Big]_{t-7}^{t+2} = \frac{e^{-t-2} - e^{-t+7}}{-1}$

$= e^{-t+7} - e^{-t-2}$

$= e^{-t}(e^{-7} - e^{-2})$



(5) $x(t) = 1 + s(t)\sin^2(5\pi t)$

$$R_x(t,\tau) = E[x(t)x(t-\tau)] = E\left[\left(1 + s(t)\sin^2(5\pi t)\right)\left(1 + s(t-\tau)\sin^2(5\pi(t-\tau)\right.\right.$$

$$= E\left[1 + s(t-\tau)\sin^2(5\pi(t-\tau)) + s(t)\sin^2 5\pi t + s(t)s(t-\tau)\sin^2 5\pi t \sin^2(5\pi(t-\tau)\right.$$

$$= 1 + \underbrace{E[s(t-\tau)]}_{0}\sin^2(5\pi(t-\tau)) + \underbrace{E[s(t)]}_{0}\sin^2 5\pi t$$

(La media è nulla perché non c'è $\delta(f)$ nello spettro di potenza)

$$+ \underbrace{E[s(t)s(t-\tau)]}_{R_s(\tau)}\sin^2 5\pi t \sin^2(5\pi(t-\tau))$$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$= 1 + R_s(\tau)\left(\frac{1}{2} - \frac{1}{2}\cos 10\pi t\right)\left(\frac{1}{2} - \frac{1}{2}\cos 10\pi(t-\tau)\right)$$

$$= 1 + R_s(\tau)\left(\frac{1}{4} - \frac{1}{4}\cos 10\pi(t-\tau) - \frac{1}{4}\cos 10\pi t + \frac{1}{4}\cos 10\pi t \cos 10\pi(t-\tau)\right)$$

$$= 1 + R_s(\tau)\left(\frac{1}{4} - \underline{\frac{1}{4}\cos 10\pi(t-\tau)} - \underline{\frac{1}{4}\cos 10\pi t} + \frac{1}{8}\cos 10\pi\tau + \underline{\frac{1}{8}\cos 10\pi(2t-\tau)}\right)$$

Processo ciclostazionario (Autocorrelazione periodica in t con periodo $T = \frac{1}{5}$). Valutiamo pertanto la autocorrelazione mediata nel periodo (i termini periodici si annullano)

$$\overline{R_x}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_x(t,\tau)\,dt = 1 + R_s(\tau)\left(\frac{1}{4} + \frac{1}{8}\cos 10\pi\tau\right)$$

$$= 1 + \frac{1}{4}R_s(\tau) + \frac{1}{8}R_s(\tau)\cos 10\pi\tau$$

$$\overline{P_x(f)} = \mathcal{F}\left[\overline{R_x}(\tau)\right] = \delta(f) + \frac{1}{4}P_s(f) + \frac{1}{16}P_s(f-5) + \frac{1}{16}P_s(f+5)$$